Matching and point processes: exercises part II

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Ratings: **.** routine (but good practice); ***** more interesting or challenging;
****** research problem (let me know if you have ideas)

2-color matching, continued

1. Show that if $(X_n)_{n\geq 1}$ are identically distributed with finite mean then $X_n/n \to 0$ a.s.

2. Show that if $X_n \xrightarrow{L^1} c$ (where c is a constant) and $X_n \stackrel{d}{=} Y_n$, then $Y_n \xrightarrow{L^1} c$. (Questions 1 and 2 were used in the lower bound for 2-color matching in dimension 2).

3* Let μ be a probability measure on the positive integers. Starting with a Poisson process of intensity 1 on $\mathbb{R}^d$, put a random number of red particles at each Poisson point, where the numbers are i.i.d. with law μ (thus the red particles form a compound (non-simple) point process). Generate blue particles in the same way using an independent Poisson process and another probability measure ν. If μ and ν both have mean ∞, prove that there is a translation-invariant perfect matching scheme of red particles to blue particles. (Hint: use additional randomness to split the particles into a suitable countable sequence of simple point processes, and use a matching scheme we already have).

4* Isometry-invariance can make a difference. Take a Poisson process of intensity 2 on $\mathbb{R}$, and color the points *alternately* red and blue, flipping a fair coin to decide on the color of the point closest to 0. This gives (dependent) jointly translation-invariant and ergodic point processes $\mathcal{R}$ and $\mathcal{B}$. Show that there is a translation-invariant matching of $\mathcal{R}$ and $\mathcal{B}$ in which X has exponential tails, but any translation-ergodic, isometry-invariant matching has $\mathbb{E}^* X = \infty$.

5. If U and V are independent Poisson random variables with common mean μ, show that

$$\mathbb{P}\Big(\frac{U-V}{U+V} > a\Big) \leq \exp -\mu(a^2 + O(a^4))$$

uniformly in μ as $a \downarrow 0$. (This is an ingredient in the exponential tail bound for the allocation in $d \geq 3$.)

Stable matching

6. Show that in a stable two-color matching M in $d = 1$, edges cannot "cross", i.e. there do not exist points $w < x < y < z$ with edges from w to y and from x to z.

7. We saw that adding a blue point makes the stable matching better for red points. Show that it makes it worse for the existing blue points. That is, for $R, B' \subset \mathbb{R}^d$ disjoint with $B' = B \cup \{b'\}$ and $R \cup B$ discrete and non-equidistant with no infinite descending chains, show that the stable matchings m of R, B and m' of R, B' satisfy $|b - m'(b)| \geq |b - m(b)|$ for all $b \in B$.

8* Find disjoint sets $R, B \subset \mathbb{R}$, with $R \cup B$ countable and non-equidistant, such that:
(a) there is no stable partial matching of R to B;
(b) $R \cup B$ is discrete, and there is more than one stable perfect matching.
What does the iterated mutually nearest neighbor matching algorithm do in these cases?

9* In $\mathbb{R}$, suppose that a red-blue pair may be matched only if the red point lies to the left of the blue point, and that points prefer closer partners, provided they are on the correct side (left or right). Define one-sided-stable matching accordingly. Show that, if the red and blue points form a locally finite non-equidistant set, there is a unique one-sided-stable matching, and the partner of a red point x is the blue point located at

$$\inf \big\{ y > x : \mathcal{R}([x, y]) = \mathcal{B}([x, y]) \big\}.$$

Deduce that in the case of independent Poisson processes, X can be expressed exactly in terms of a random walk.

10* For the alternating-color processes in Exercise 4, show that the stable matching has $\mathbb{P}^*(X > r) < c/r$ for some c and all $r > 0$.

11. For any non-equidistant discrete set of points in $\mathbb{R}^d$ with no descending chains, show that there is a unique one-color stable partial matching. In the case of a translation-invariant point process, show that it is a.s. a perfect matching.

12* For *any* translation-invariant non-equidistant simple point process on $\mathbb{R}^d$ with intensity 1 and no descending chains, prove that for the one-color stable matching, $\mathbb{P}^*(X > r) < c/r^d$ for all $r > 0$, where c depends only on d. (Hint: how close can two r-bad points be?)

13. Show that, if the simple point process Π is insertion-tolerant, then so is $\Pi + \delta_U$, where U is uniformly distributed on any given set of finite volume. (Hint: show first that $\Pi + \delta_x$ is insertion-tolerant for almost every deterministic x.)

14. Give examples of translation-invariant simple point processes of finite intensity with the following properties:
(a) insertion-tolerant but not deletion-tolerant;
(b) deletion-tolerant but not insertion-tolerant;
(c) deletion-tolerant but $\Pi^* \not\prec \Pi + \delta_0$.

15** Improve the tail bounds for the stable matching of two independent Poisson processes in $d \geq 2$. Is the correct power $d/2$?

Further topics

16. Consider the *asymmetric* matching rule in which blue points are only allowed to match to red points, but red points may match to points of either color. Suppose that red and blue points occur as independent Poisson processes of respective intensities ρ and β in $\mathbb{R}^d$. Show that if $\rho < \beta$ there is no perfect invariant matching, while if $\rho = \beta$ the possible tail behavior of X is the same as for 2-color matching.

17* In the asymmetric model of Question 16, with $\rho > \beta$ and $d = 1$, show that there are invariant matchings where X has exponential tail.

18* Define and construct the *stable* matching subject to the asymmetric rule of Question 16. Show that if $\rho < \beta(1+\epsilon)$ for some $\epsilon = \epsilon(d) > 0$ then there are unmatched blue points.

19** In the model of the last question, do there exist any ρ, β, d for which the matching is perfect?

20** Let $\mathcal{R}, \mathcal{B}$ be independent Poisson processes of intensity 1 in $\mathbb{R}^2$. Does there exist a translation-invariant matching scheme in which the line segments joining matched pairs do not cross?

21** In the setting of Question 20, does there exist a matching (invariant or otherwise) in which every finite set of edges minimizes the total length among all matchings of their incident points?