Multi-dimensional Percolation

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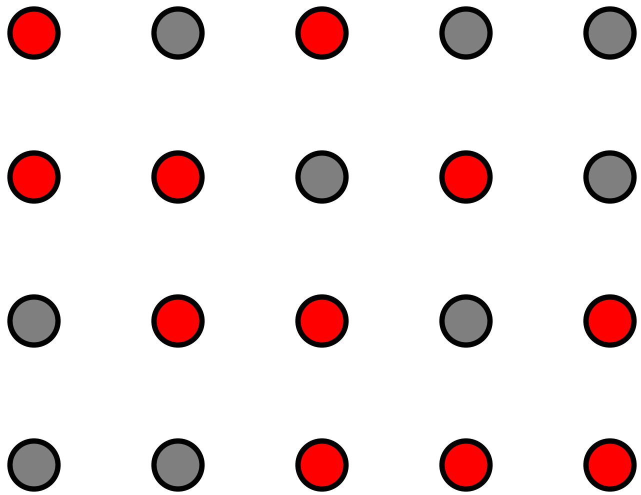
- **Lipschitz percolation** (N. Dirr, P. W. Dondl, G. R. Grimmett, A. E. Holroyd, M. Scheutzow; EJP, arXiv)
- **Lattice embeddings in percolation** (G. R. Grimmett, A. E. Holroyd; arXiv)

Site percolation: sites of Z^d independently

$$\begin{cases} \text{open} & \text{with prob. } p \\ \text{closed} & \text{with prob. } 1-p \end{cases}$$



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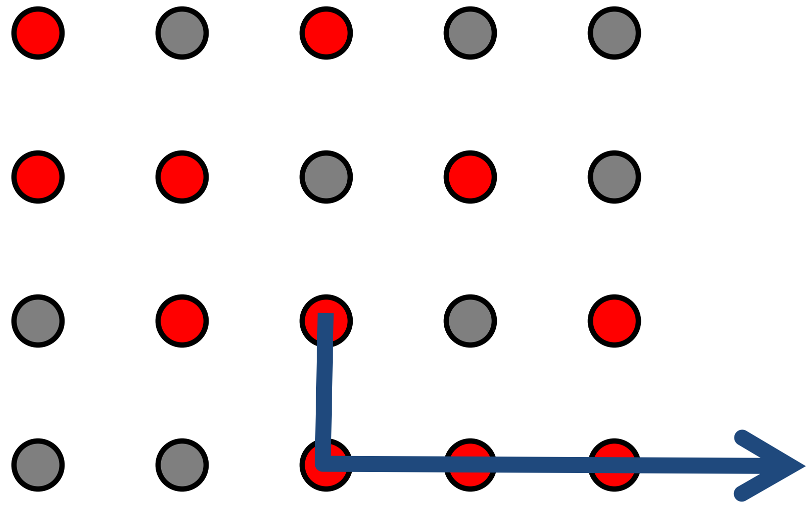

$p_c := \inf\{p: \exists$ infinite open path a.s.$\}$

$\in (0,1)$ for $d \geq 2$;

$= 1$ for $d = 1$

Site percolation: sites of Z^d independently

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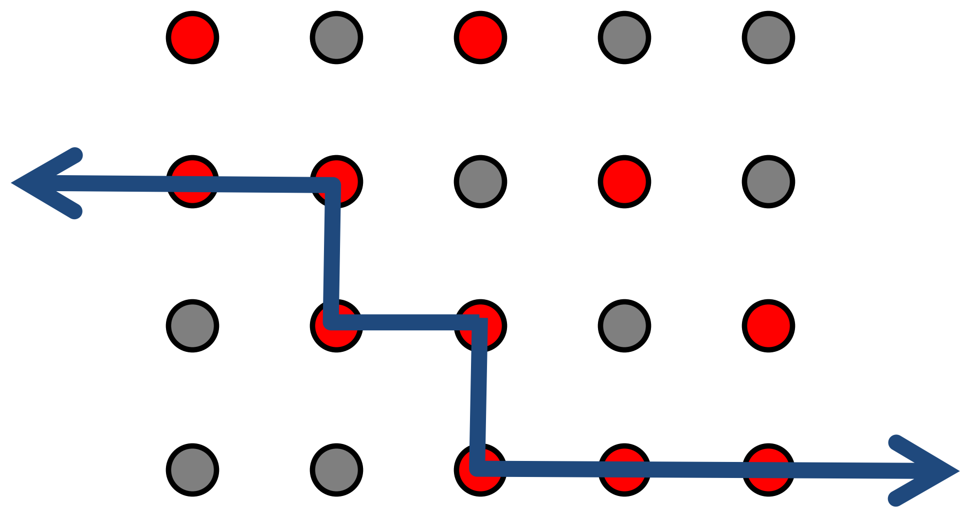

$p_c = \inf\{p: \exists \text{ bi-infinite open path a.s.}\}$

Site percolation: sites of Z^d independently

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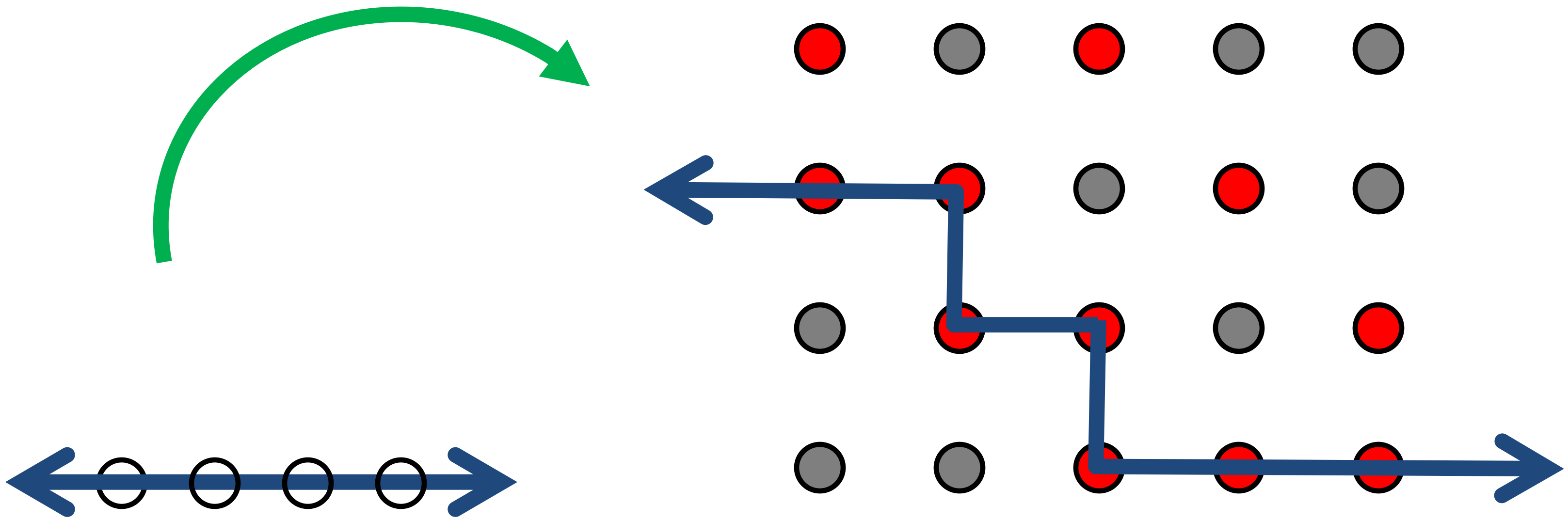


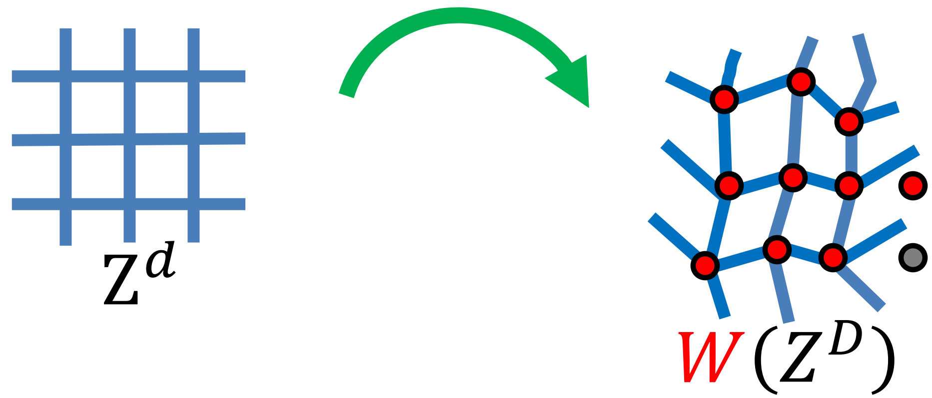
$p_c=\inf\{p: \exists$ 1-Lipschitz injection $Z \to W(Z^d)$ a.s.$\}$

$|x-y|_1 = 1 \Rightarrow |f(x)-f(y)|_1 \leq 1$

{open sites}

Define

$p_c(d,D)$:= inf{p: ∃ 1-Lip. inj. $Z^d \to W(Z^D)$ a.s.}



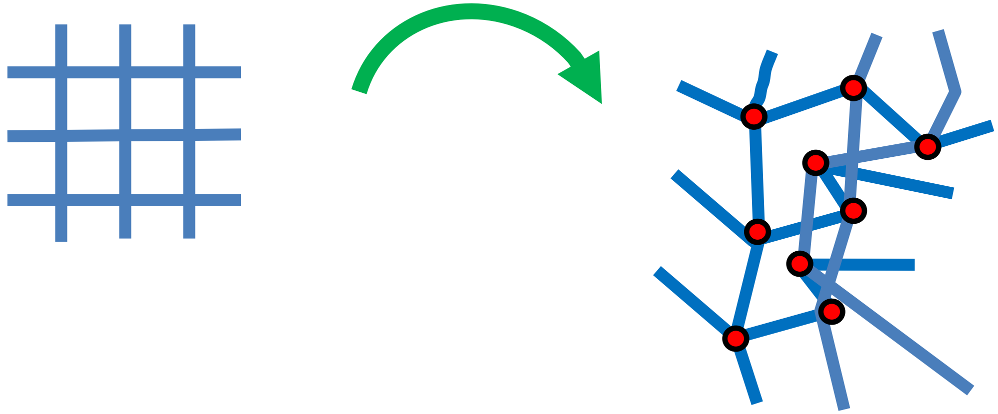
As above, $p_c(1, D) = p_c(Z^D)$

Clearly, $p_c(d, D) = 1$ for $d > D$

Theorem (GH): $p_c(d, D) = 1$ for $d \geq 2$

So define $|x-y|_1 = 1 \Rightarrow |f(x)-f(y)|_1 \le M$

$p_c(d,D,M) := \inf\{p : \exists\ M\text{-Lip. inj. } Z^d \to W(Z^D) \text{ a.s.}\}$



Previous theorem: $p_c(d,D,1) = 1$ for $d \ge 2$

$p_c(d,D,M) > 0$ for all d,D,M

<u>Main question</u>: $p_c(d,D,M) \begin{cases} < 1 \\ = 1 \end{cases}$?

Theorem (DDGHS): $p_c(d, D, 2) < 1$ for $d < D$

Clearly, $p_c(d, D, M) = 1$ for $d > D$, all M

Theorem (GH): $p_c(d, d, M) = 1$ for all M

So, writing

$M_c(d, D) := \min\{M \colon p_c(d, D, M) < 1\} =$

$\min\{M \colon \exists M\text{–Lip inj } Z^d \to W(Z^D), \text{for some } p < 1\}$

D

d

$M_c(d, D)$	1	2	3	4	5
1	∞	1	1	1	1
2	∞	∞	2	2	2
3	∞	∞	∞	2	2
4	∞	∞	∞	∞	2
5	∞	∞	∞	∞	∞

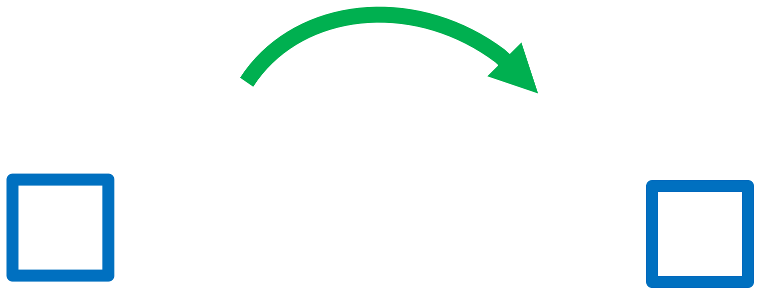
Some Proofs...

Theorem: $p_c(d, D, 1) = 1$ for $d \geq 2$

Theorem: $p_c(d, D, 2) < 1$ for $d < D$

Theorem: $p_c(d, d, M) = 1$ for all M

Easy case: d=2, D=3, M=1



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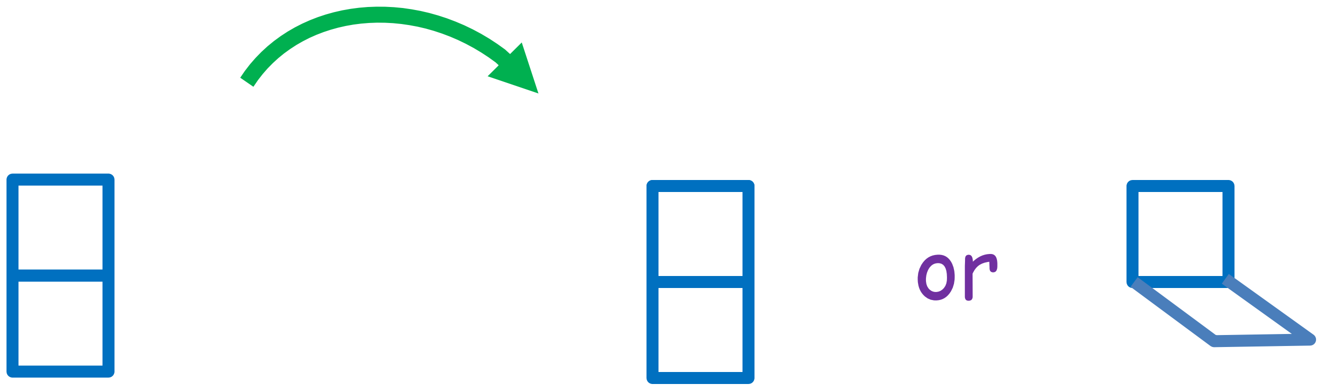

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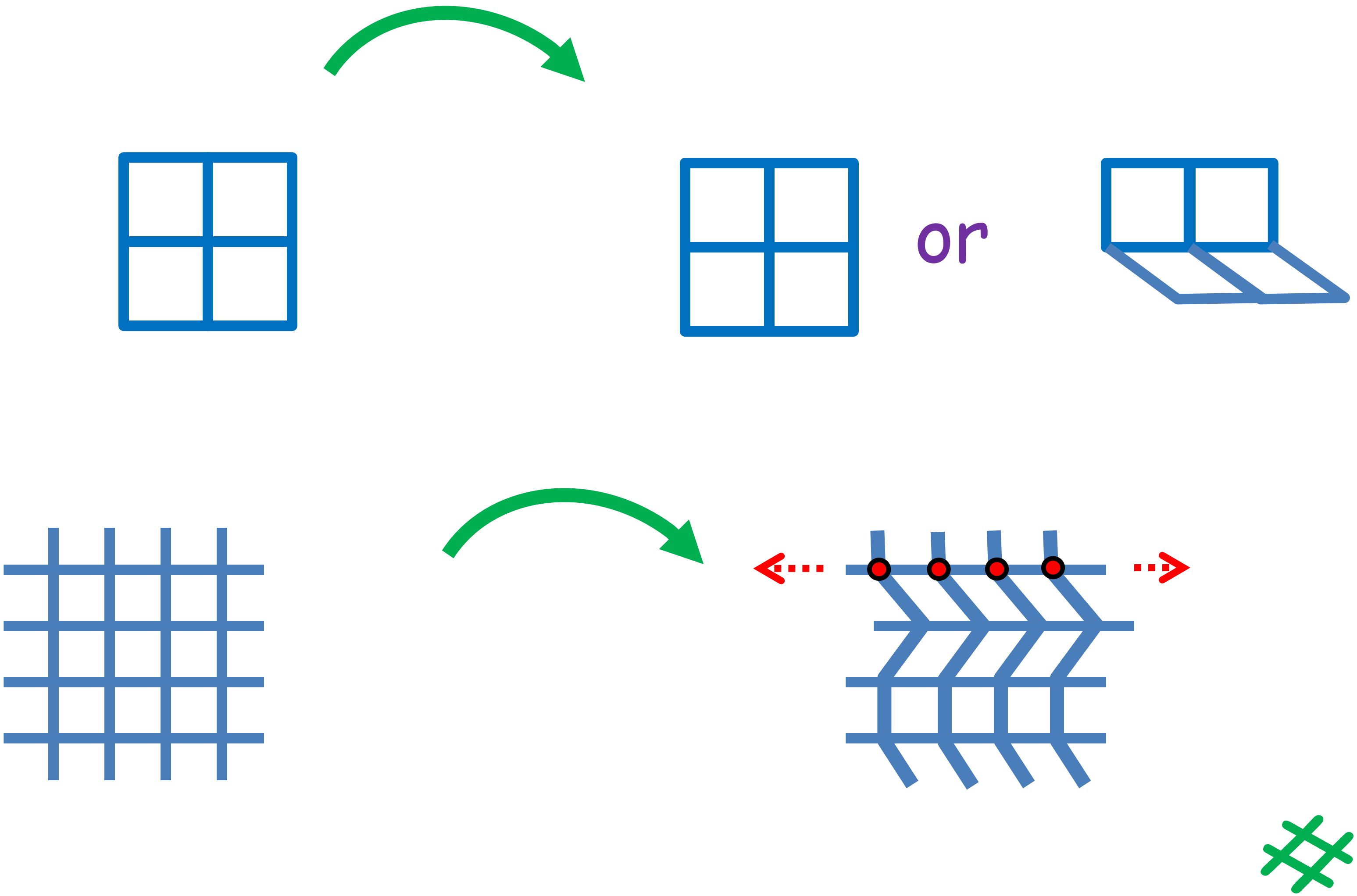
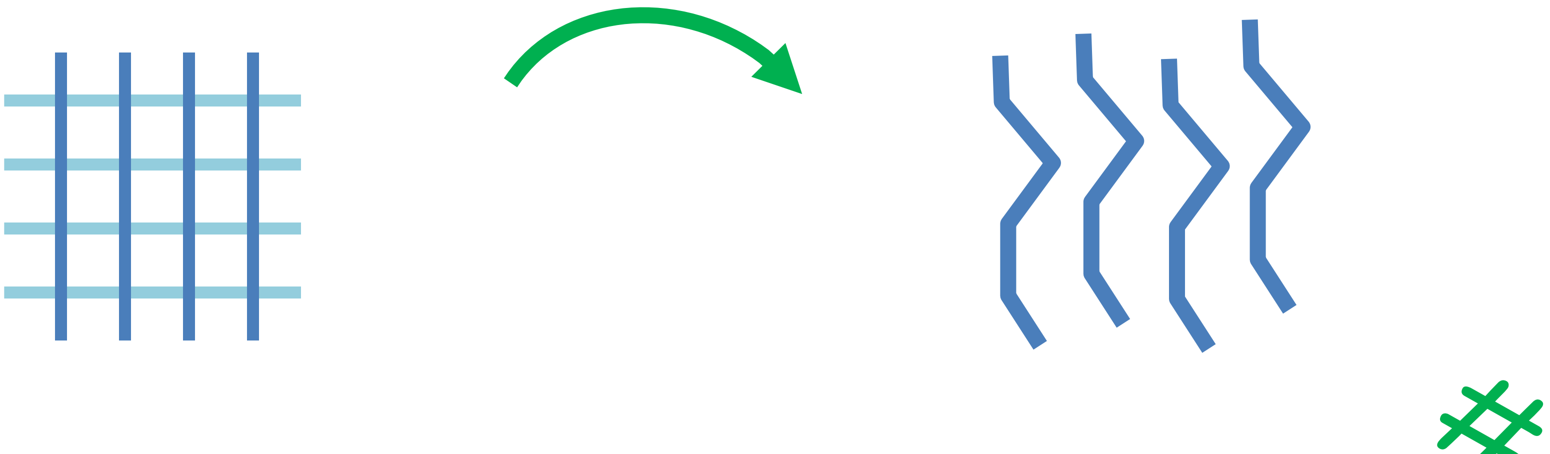


General case: d=2, M=1



Theorem: $p_c(d, D, 1) = 1$ for $d \geq 2$

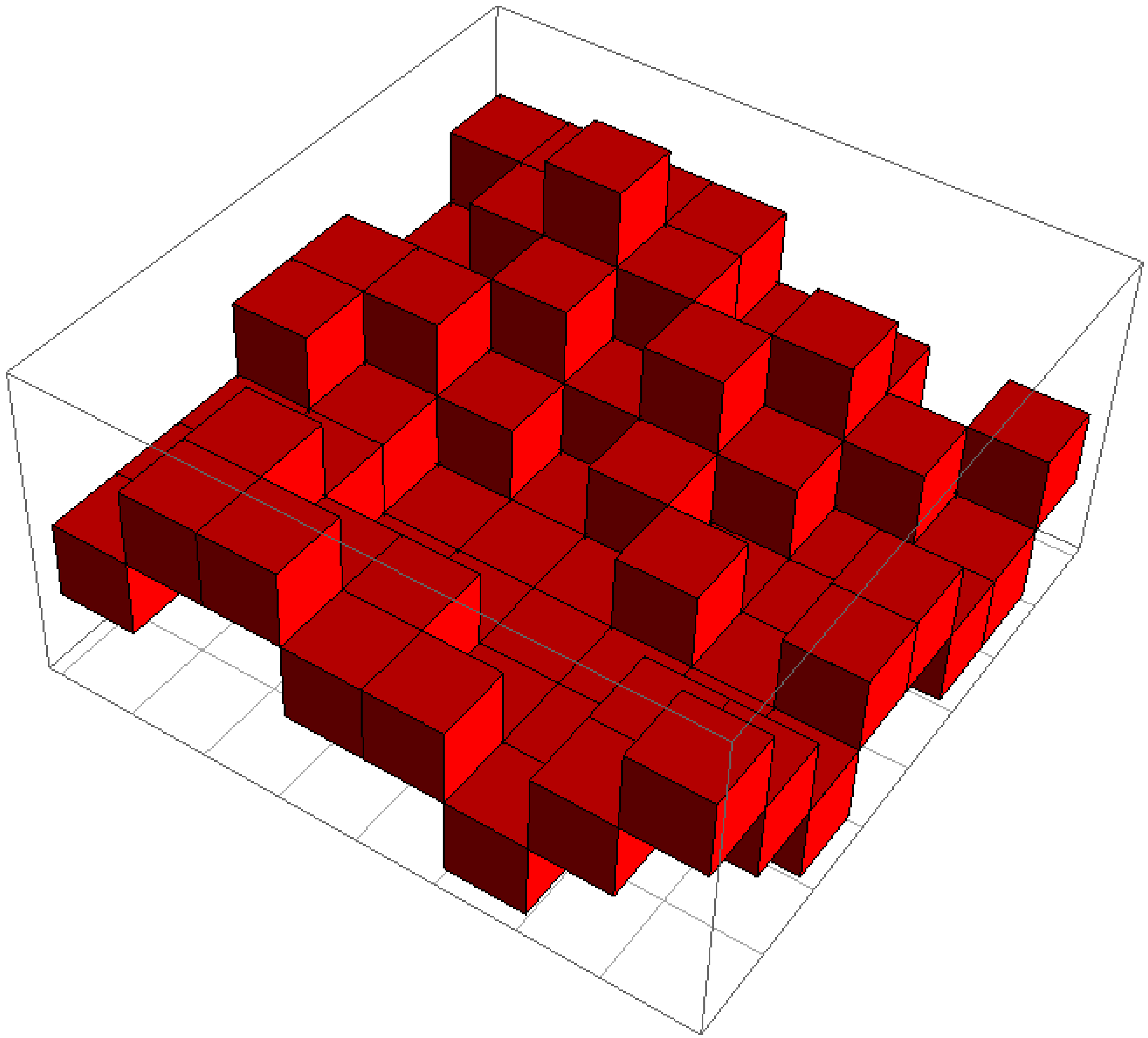
Theorem: $p_c(d, D, 2) < 1$ for $d < D$

Theorem: $p_c(d, d, M) = 1$ for all M

Stronger result (DDGHS): For $p > 1-(2d)^{-2}$,

a.s. ∃ 1-Lip map $F\colon Z^{d-1} \to Z_+$

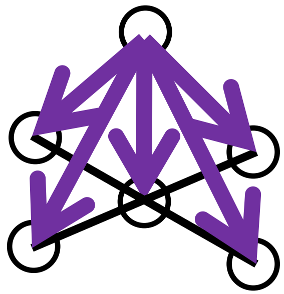
such that (u,F(u)) open $\forall u$.



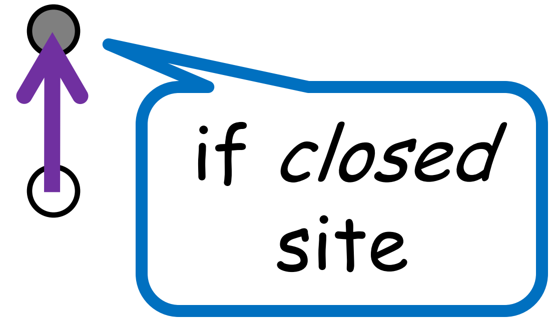
Proof:

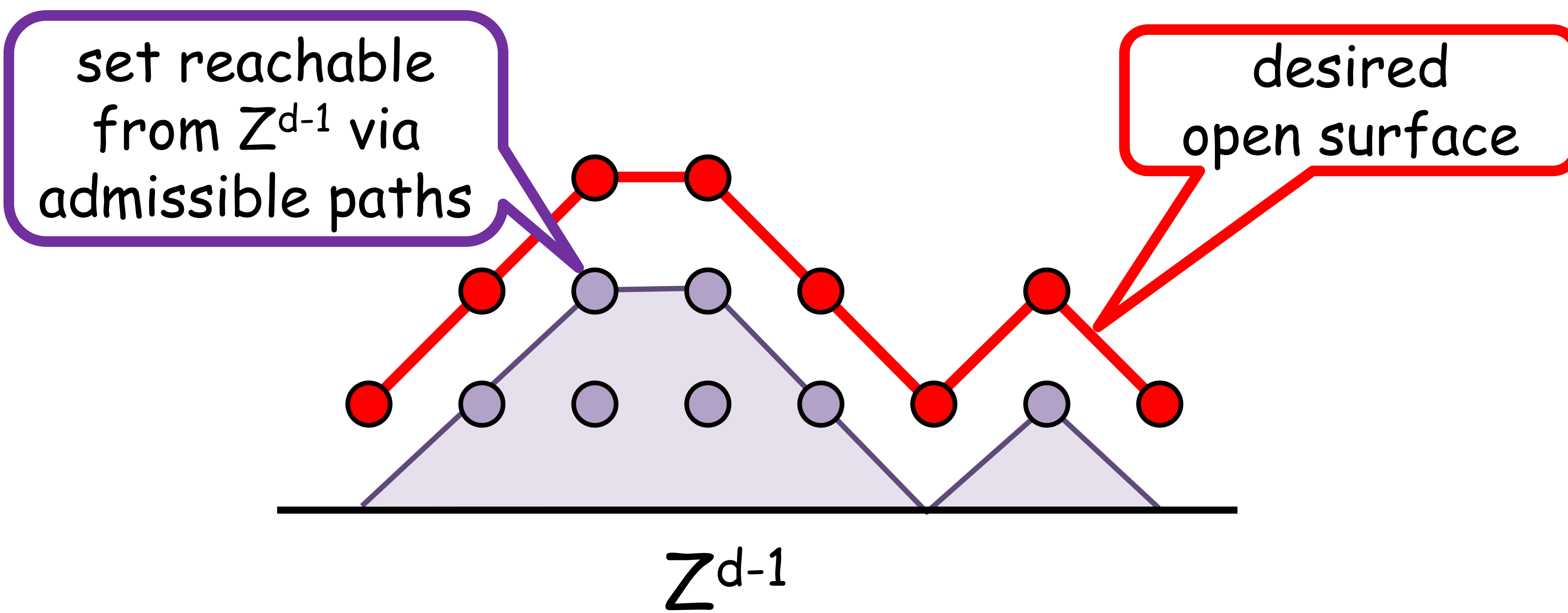
Admissible path:

steps:



and





Remains to show:
some sites not reachable from Z^{d-1}.

E # {admiss. paths from Z^{d-1} to +ve d-axis}

$$\leq \sum_{U \geq D \geq 0} (2d)^{U+D} q^U$$

q=1-p

$$\leq \frac{1}{(1-2dq)(1-(2d)^2 q)}$$

$$< \infty$$

if $q < (2d)^{-2}$

□

(bound can be improved to $p > [8(d-1)]^{-1}$)

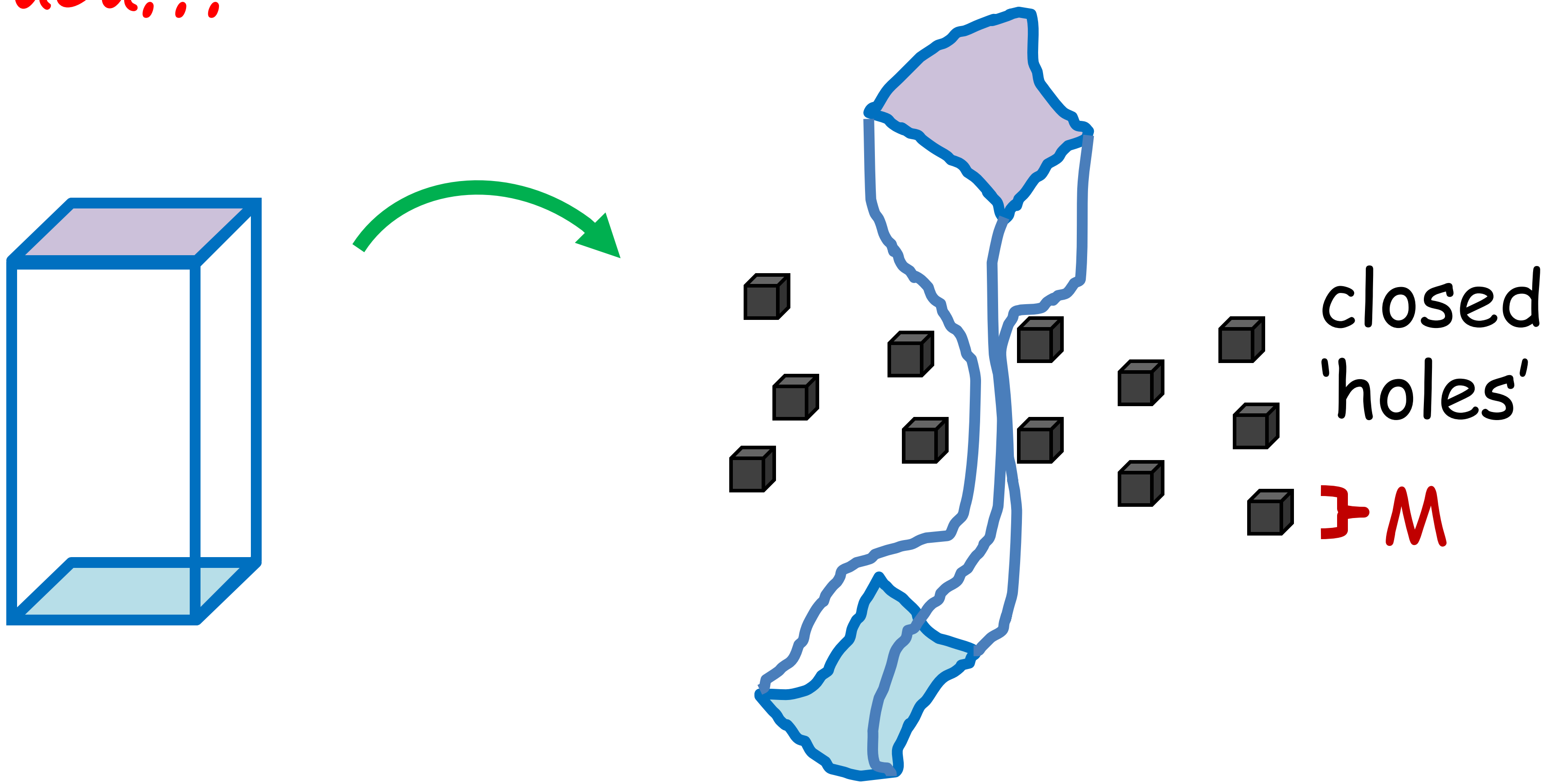
Theorem: $p_c(d, D, 1) = 1$ for $d \geq 2$

Theorem: $p_c(d, D, 2) < 1$ for $d < D$

Theorem: $p_c(d, d, M) = 1$ for all M

Need to show: no M-Lip inj. $\mathbb{Z}^d \longrightarrow W(\mathbb{Z}^d)$

Idea...



Lemma 1: *-lattice $(V,E)=(Z^d, |\cdot|_\infty=1)$.

Any coloring χ: $[1,n]^{d-1} \times [1,m] \to \{\pm\infty, 1, 2, \ldots, d-1\}$.

map

If:

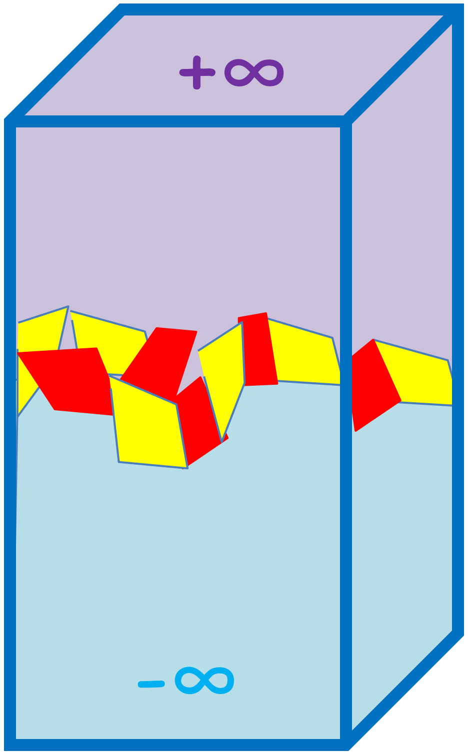
1) $-\infty$ and $+\infty$ not adjacent

2) bottom face colored $-\infty$

3) top face colored $+\infty$,

then:

$\exists$ a cluster of color $\in \{1,\ldots,d-1\}$ and volume $\geq n$.



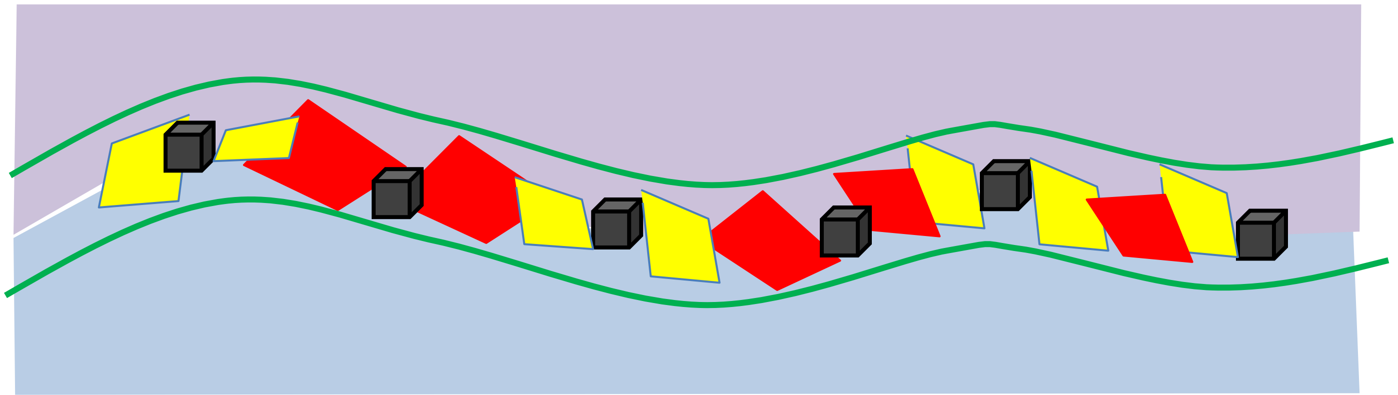
Lemma 2: $(V,E)=(W(Z^d), |\cdot|_\infty \le J)$.

a.s. $\exists$ coloring $\lambda: W(Z^d) \to \{\pm\infty, 1, 2, \ldots, d-1\}$

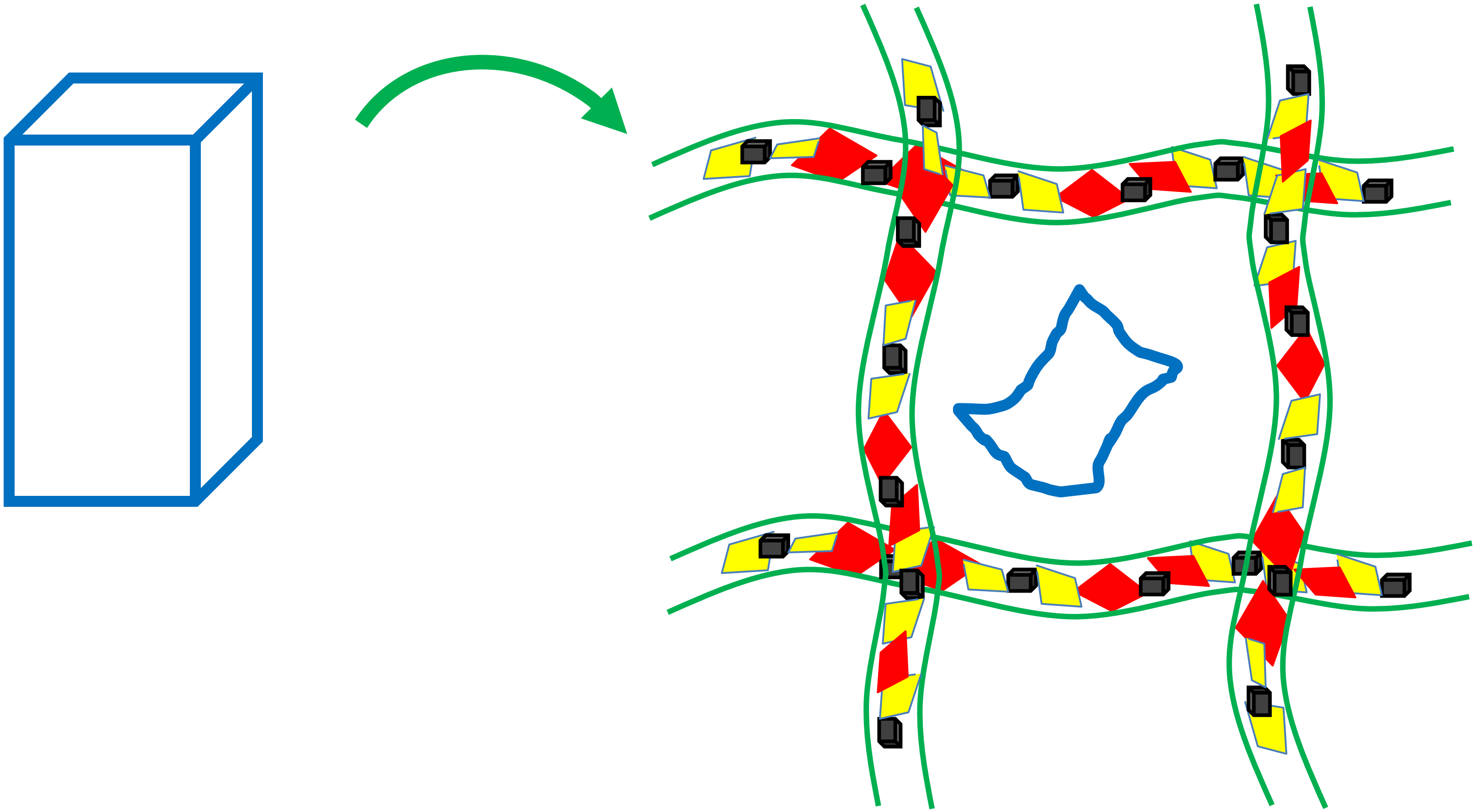
and ϵ-Lip $g: R^{d-1} \to R_+$ s.t.:

$\forall$ p,J,d,ϵ $\exists$ C,K

1) $-\infty$ and $+\infty$ not adjacent
2) $\{(x,z): z < g(x)-C\}$ colored $-\infty$
3) $\{(x,z): z > g(x)+C\}$ colored $+\infty$
4) clusters of colors $\in \{1,2,\ldots,d-1\}$ have volume $\le K$



Proof of Theorem (from Lemmas)



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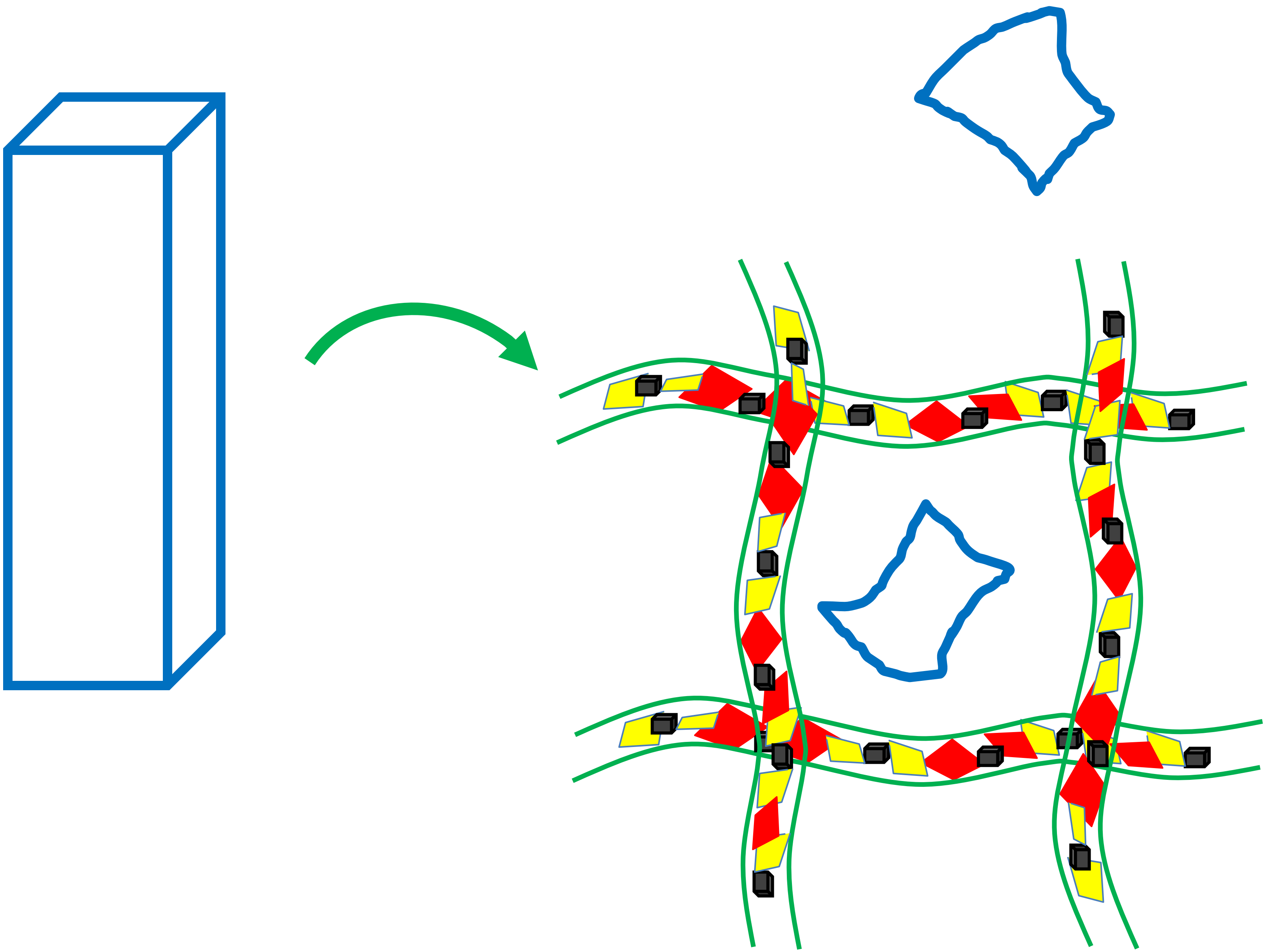

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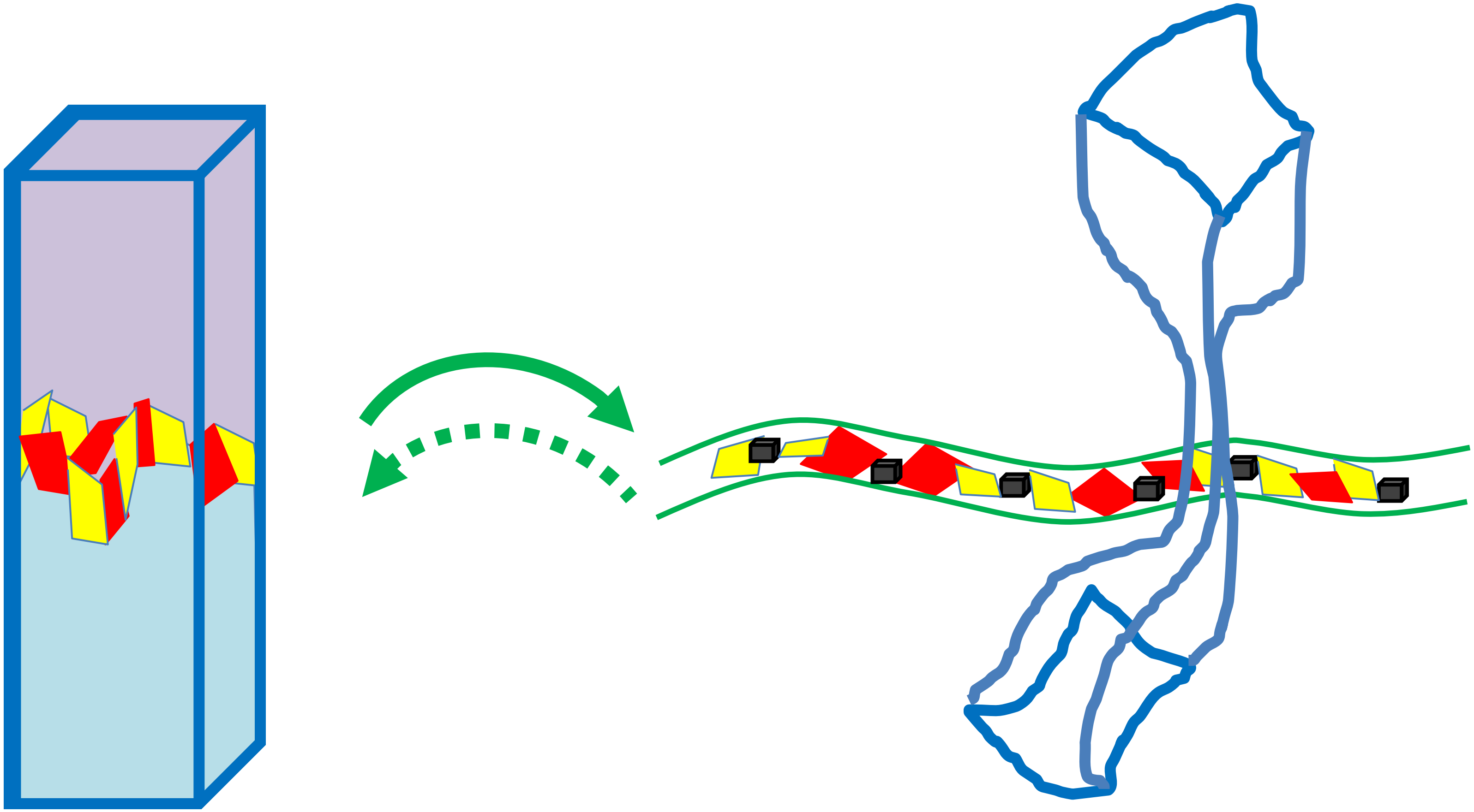
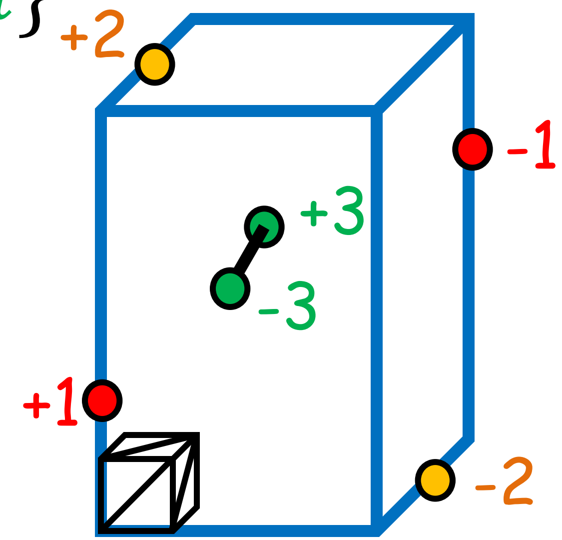


Tucker's Lemma (for the cuboid)

$V = ([0, t_1] \times \cdots \times [0, t_d]) \cap Z^d \qquad t \in Z^d$

$E = \{(u, v):\ u_i \leq v_i \leq u_i + 1 \ \ \forall i\}$

Coloring $\kappa: V \to \{\pm 1, \pm 2, \ldots, \pm d\}$



If antipodal vertices have opposite colors:

$x, y \in \partial V, \qquad x + y = t$

$\Rightarrow \ \kappa(x) = -\kappa(y),$

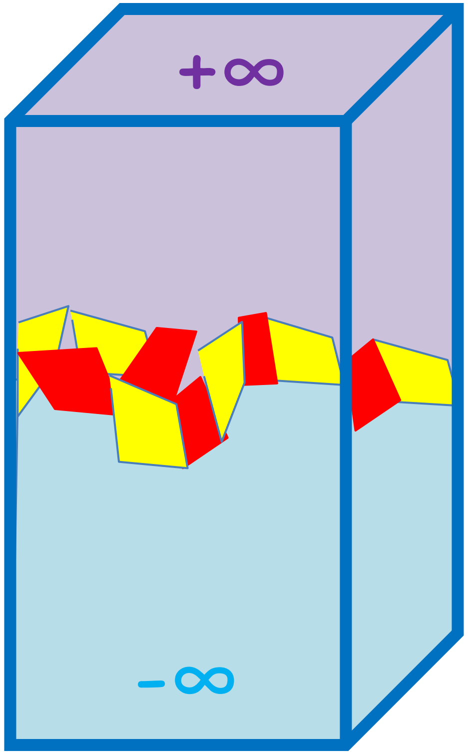
then ∃ adjacent vertices with opposite colors: $\quad u, v \in V: \ \ \kappa(u) = -\kappa(v).$

Proof of Lemma 1

χ: $[1,n]^{d-1} \times [1,m] \to \{\pm\infty, 1, 2, \ldots, d-1\}$.

Suppose:

1) $-\infty$ and $+\infty$ not adjacent
2) bottom face colored $-\infty$
3) top face colored $+\infty$,
4) All clusters of colors $\in \{1,\ldots,d-1\}$ have *diameter* $\leq$ n



Proof of Lemma 1

χ: $[1,n]^{d-1}$x$[1,m] \rightarrow \{\pm\infty, 1, 2, \ldots, d-1\}$.

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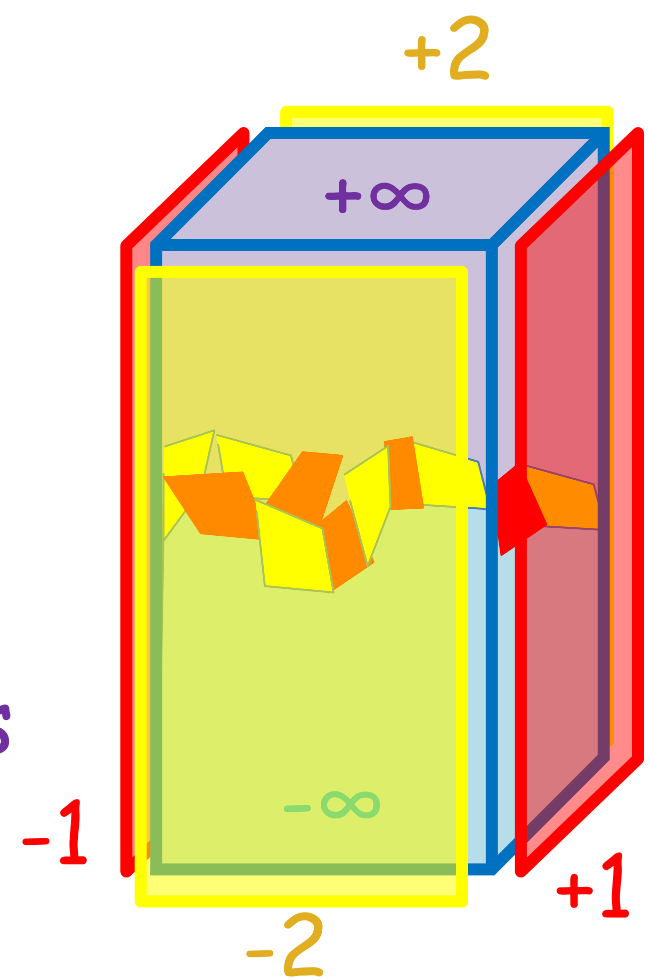
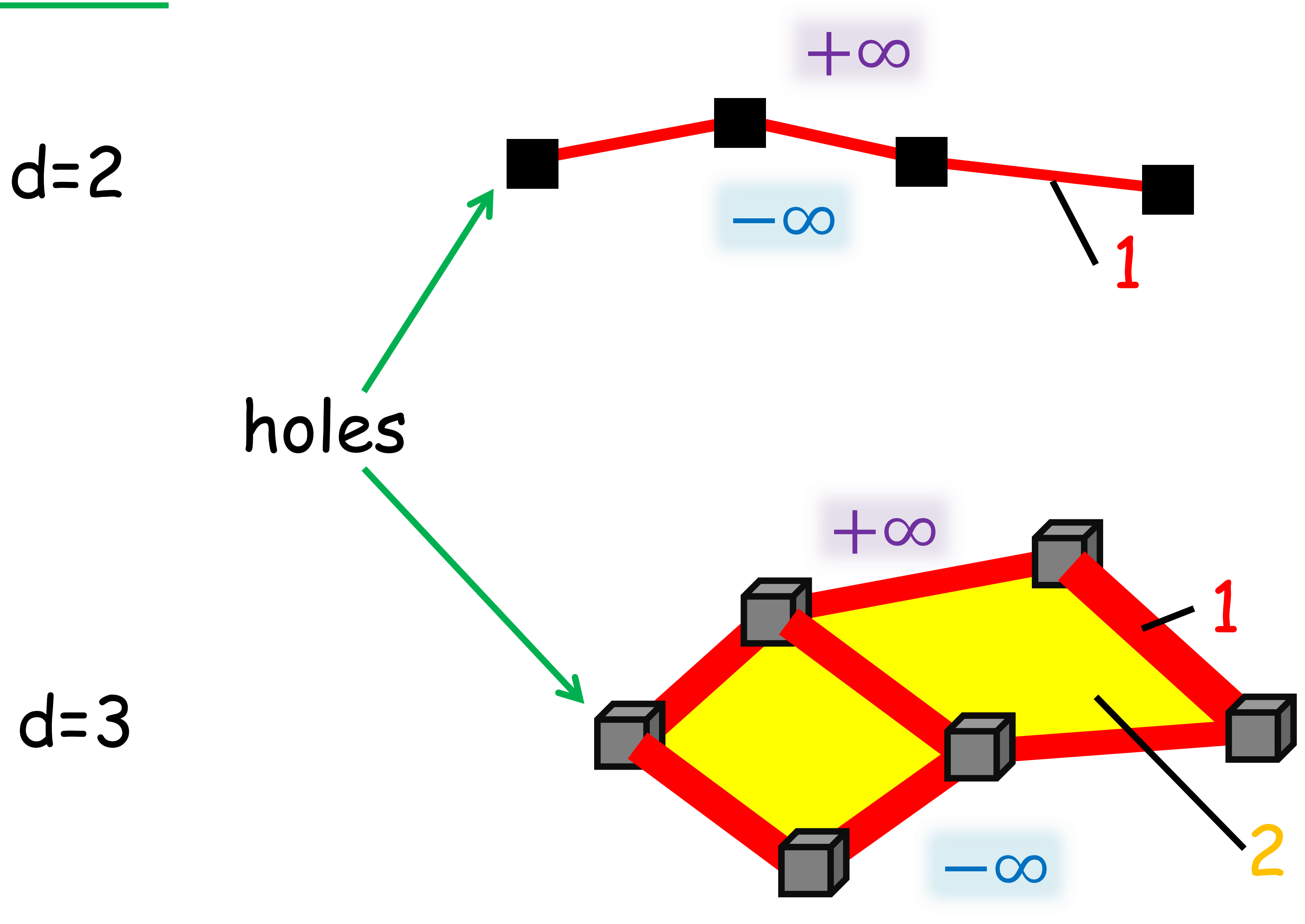


- Add faces with signed colors
- Recolor j-clusters touching -j face $\rightarrow$ color -j
- Apply Tucker's Lemma

#

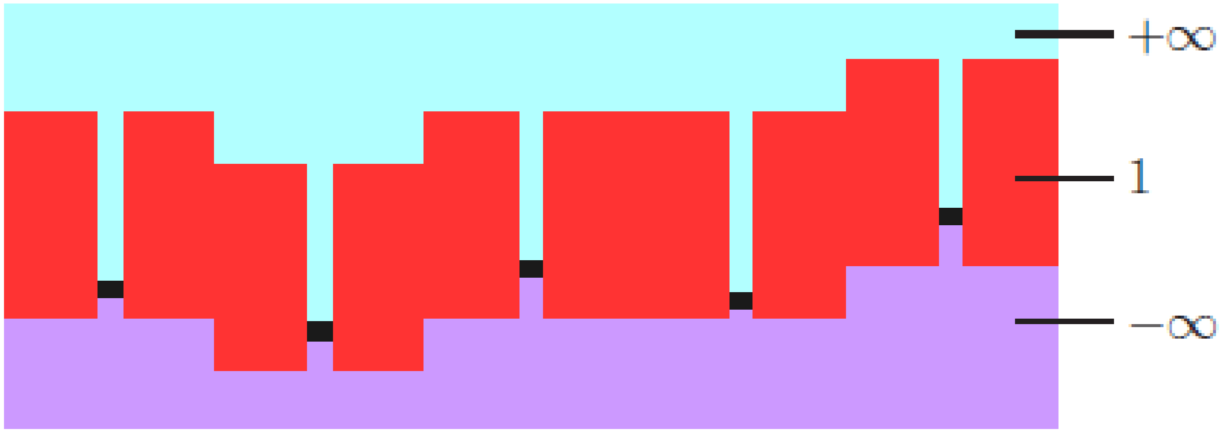
Proof of Lemma 2

Natural approach:

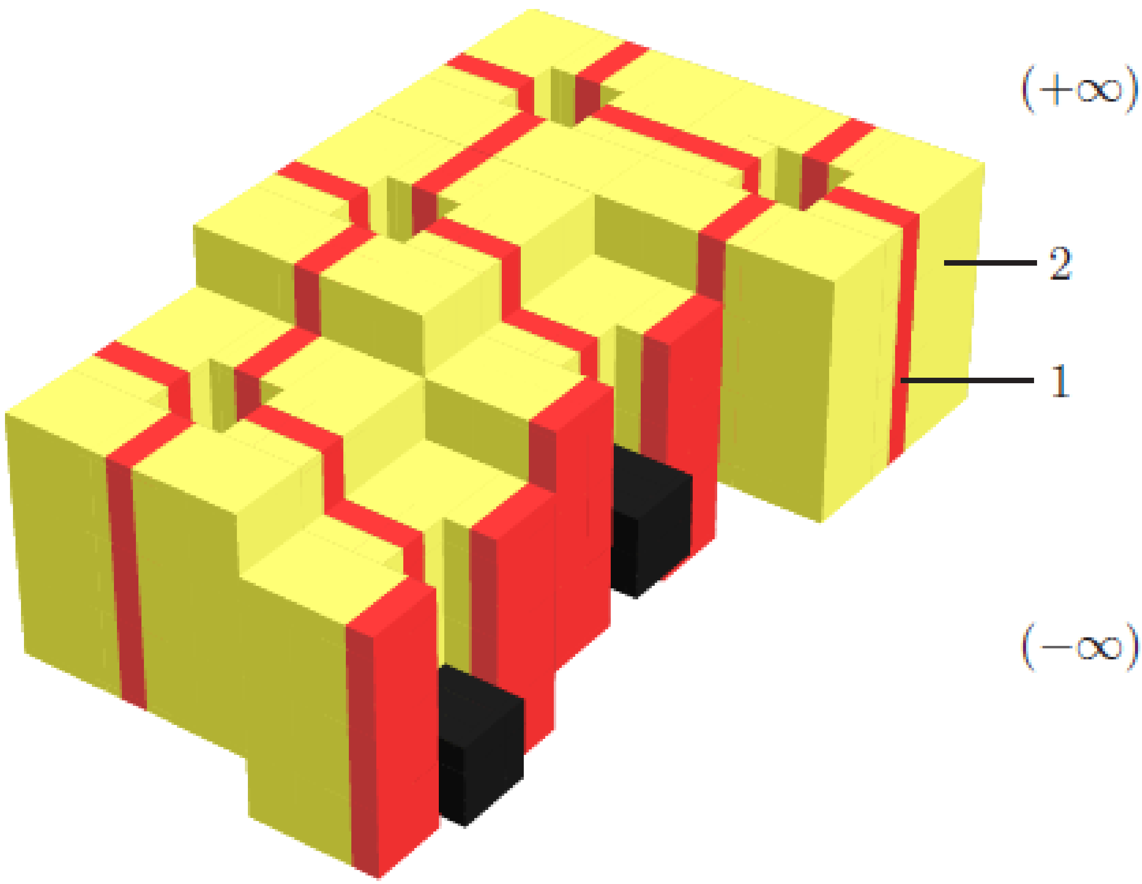


Proof of Lemma 2

Actual construction:

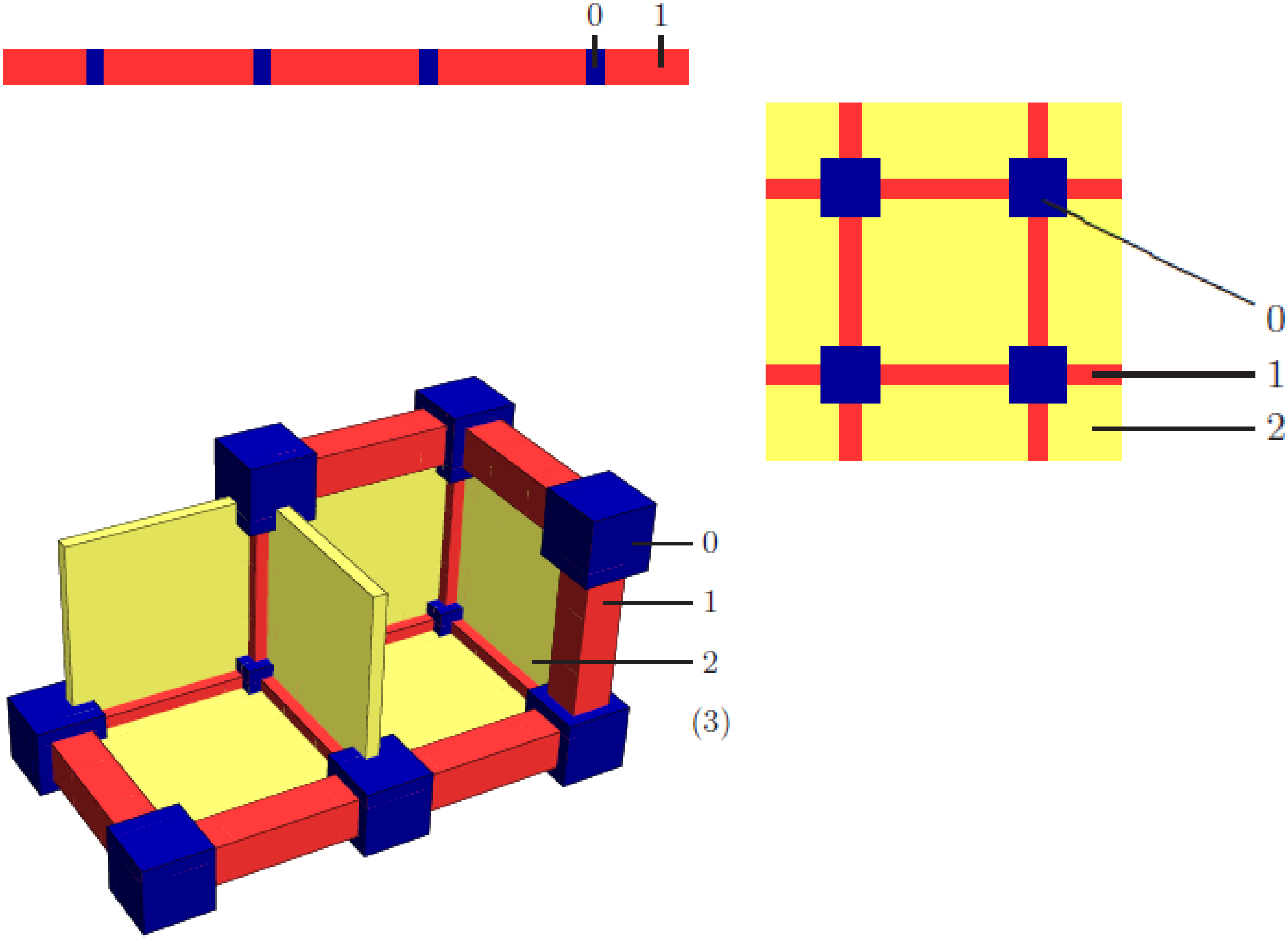


d=2



d=3

Use 2-Lip surface result, and...



(3)

Open Questions

- Quantitative versions: $[1,n]^d \to W([1,N]^D)$
- Other graphs: $V(G) \to W(V(G))$
- Interpolate between 1-Lip and 2-Lip
- Embedding:

$$\exists ?\ \ \eta \in \{0,1\}^{Z^d}: \quad \eta \to \omega_p(Z^d)$$