SYMmetric 1–Dependent Colorings of the Integers

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Abstract. In a recent paper by the same authors, we constructed a stationary 1–dependent 4–coloring of the integers that is invariant under permutations of the colors. This was the first stationary k–dependent q–coloring for any k and q. When the analogous construction is carried out for q > 4 colors, the resulting process is not k–dependent for any k. We construct here a process that is symmetric in the colors and 1–dependent for every q ≥ 4. The construction uses a recursion involving Chebyshev polynomials evaluated at \( \sqrt{q}/2 \).

1. Introduction

By a (proper) q–coloring of the integers, we mean a sequence \((X_i : i \in \mathbb{Z})\) of \([q]\)–valued random variables satisfying \(X_i \neq X_{i+1}\) for all \(i\) (where \([q] := \{1, \ldots, q\}\)). The coloring is said be stationary if the (joint) distribution of \((X_i : i \in \mathbb{Z})\) agrees with that of \((X_{i+1} : i \in \mathbb{Z})\), and \(k\)–dependent if the families \((X_i : i \leq m)\) and \((X_i : i > m + k)\) are independent of each other for each \(m\). In [2], we gave a construction of a stationary 1–dependent 4–coloring of the integers that is invariant under permutations of the colors. When the same construction is carried out for \(q > 4\) colors, the resulting distribution is not \(k\)–dependent for any \(k\). Of course, the 1–dependent 4–coloring is also a 1–dependent \(q\)–coloring for every \(q > 4\), and one may obtain other 1–dependent \(q\)–colorings by splitting a color into further colors using an independent source of randomness. However, these colorings are not symmetric in the colors. We give here a modification of the process of [2] that is symmetric in the colors and 1–dependent for every \(q ≥ 4\). Here is our main result.

Theorem 1. For each integer \(q ≥ 4\), there exists a stationary 1–dependent \(q\)–coloring of the integers that is invariant in law under permutations of the colors and under the reflection \((X_i : i \in \mathbb{Z}) \mapsto (X_{-i} : i \in \mathbb{Z})\).

Our construction is given in the next section. Sections 3 and 4 provide some preliminary results and the proof of Theorem 1 respectively.

2. The construction

For \(x = (x_1, x_2, \ldots, x_n) \in [q]^n\), we will write \(P(x) = \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)\). To motivate the construction, we begin by noting that the finite-dimensional distributions \(P\) of the 4–coloring in [2] are defined recursively by \(P(\emptyset) = 1\) and

\[
P(x) = \frac{1}{2(n+1)} \sum_{i=1}^{n} P(\tilde{x}_i)
\]
for proper $x \in [4]^n$, where $\hat{x}_i$ is obtained from $x$ by deleting the $i$th entry in $x$. Of course, even if $x$ is proper, $\hat{x}_i$ may not be. So the definition is completed by setting $P(x) = 0$ for $x$’s that are not proper.

For general $q \geq 4$, we will now allow the coefficients in the defining sum to depend on $i$ as well as $n$. Considering many special cases, and the constraints imposed by the $1$–dependence requirement, we were led to define

$$P(x) = \frac{1}{D(n+1)} \sum_{i=1}^{n} C(n-2i+1)P(\hat{x}_i)$$

for proper $x \in [4]^n$, in terms of two sequences $C$ and $D$. Again motivated by computations in special cases, we take

$$C(n) = T_n(\sqrt{q}/2), \quad n \geq 0;$$
$$D(n) = \sqrt{q} U_{n-1}(\sqrt{q}/2), \quad n \geq 1,$$

where $T_n$ and $U_n$ are the Chebyshev polynomials of the first and second kind respectively.

There are several standard equivalent definitions of Chebyshev polynomials. One is

$$T_n(u) = \cosh(nt) \quad \text{and} \quad U_n(u) = \frac{\sinh[(n+1)t]}{\sinh(t)}, \quad \text{where } u = \cosh(t).$$

A variant definition using trigonometric functions (e.g. (22:3:3-4) of [3]) is easily seen to be equivalent by taking $t$ imaginary; the hyperbolic function version is convenient for arguments $u \geq 1$. Another definition is

$$T_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} u^{n-2k} (u^2 - 1)^k \quad \text{and} \quad U_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} u^{n-2k} (u^2 - 1)^k.$$

That this is equivalent to (3) follows from e.g. (22:3:1-2) of [3].

If $x$ is not a proper coloring, we take $P(x) = 0$ as before. We extend both sequences $C$ and $D$ to all integer arguments by taking $C(n)$ and $D(n)$ to be even and odd functions of $n$ respectively (in accordance with (3)).

Observe that $C(n)$ and $D(n)$ are strictly positive for $q \geq 4$ and $n \geq 1$, and therefore $P(x)$ is strictly positive for all proper $x$. Note also that $C(n-2i+1)/D(n+1)$ is rational; therefore so is $P(x)$. (The factors of $\sqrt{q}$ cancel). When $q = 4$ we have $C(n) = 1$ and $D(n) = 2n$, and so (2) reduces to (1) in this case. As we will see, the fact that the coefficients in (2) depend on $i$ substantially complicates the verifications of the required properties of $P$.

Here are a few examples of cylinder probabilities generated by (2).

$$P(1) = \frac{1}{q}, \quad P(12) = \frac{1}{q(q-1)}, \quad P(121) = \frac{1}{q^2(q-1)}, \quad P(123) = \frac{1}{q^2(q-2)},$$
$$P(1212) = \frac{q-3}{q^2(q-1)(q^2-3q+1)}, \quad P(1234) = \frac{1}{q^2(q^2-3q+1)}.$$
3. Preliminary results

Chebyshev polynomials satisfy a number of standard identities. They lead to identities satisfied by the sequences \( C \) and \( D \). The first three in the proposition below are examples of this. The fourth is a consequence of the third one. Before stating them, we record some values of \( C \) and \( D \) to facilitate checking computations here and later.

\[
C(0) = 1, \quad C(1) = \frac{\sqrt{q}}{2}, \quad C(2) = \frac{q - 2}{2}, \quad C(3) = \frac{\sqrt{q(q - 3)}}{2}, \quad C(4) = \frac{q^2 - 4q + 2}{2}.
\]

\[
D(0) = 0, \quad D(1) = \sqrt{q}, \quad D(2) = q, \quad D(3) = \sqrt{q(q - 1)}, \quad D(4) = q(q - 2).
\]

**Proposition 2.** For \( j, k, \ell, m, n \in \mathbb{Z} \), the following identities hold.

\[
\begin{align*}
(4) & \quad 2C(m)C(n) = C(m + n) + C(n - m). \\
(5) & \quad \frac{q - 4}{2q}D(m)D(n) = C(m + n) - C(n - m). \\
(6) & \quad 2C(m)D(n) = D(m + n) + D(n - m). \\
(7) & \quad C(j + k)D(k + \ell) = C(k)D(j + k + \ell) - C(\ell)D(j).
\end{align*}
\]

**Proof.** The first three parts are immediate consequences of (22.5-7) in [3], or 22.7.24-26 in [1], if \( m \) and \( n \) are nonnegative. None of the identities is changed by changing the sign of either \( m \) or \( n \). Therefore, they hold for all \( m \) and \( n \). Alternatively, the identities may be checked directly from (3) using the product formulae for hyperbolic functions. For (7), replace the products of \( C \)'s and \( D \)'s by sums of \( D \)'s using (6), and then use the fact that \( D \) is an odd function. \( \square \)

Next we verify some identities that involve both the sequences \( C \) and \( D \) and the measure \( P \) defined by (2). For the statement of the second part of the next result, let

\[
Q(x) = \frac{1}{D(n + 1)} \sum_{i=1}^{n} C(2i)P(\hat{x}_i) \quad \text{and} \quad Q^*(x) = \frac{1}{D(n + 1)} \sum_{i=1}^{n} C(2n - 2i + 2)P(\hat{x}_i)
\]

for \( x \in [q]^n \). The first part of Proposition 3 is needed in proving the second part, which plays a key role in the proof of consistency and \( 1 \)-dependence of \( P \). Note the similarity between the left side of (8) and the right side of (2).

**Proposition 3.** If \( n \geq 1 \), and \( x \) is a proper coloring of length \( n \), then

\[
\begin{align*}
(8) & \quad \sum_{i=1}^{n} D(n - 2i + 1)P(\hat{x}_i) = 0; \\
(9) & \quad Q(x) = Q^*(x) = P(x)C(n + 1).
\end{align*}
\]

**Proof.** For the first statement, let \( R \) be the set of proper colorings, and \( \hat{x}_A \) be obtained by deleting the entries \( x_i \) for \( i \in A \) from \( x \). The proof of (8) is by induction on \( n \), the length of \( x \). The identity is easily seen to be true if \( n \leq 2 \). Suppose that (8) is true for all \( x \) of length \( n - 1 \), and let \( x \in R \) have length \( n \). For those \( i \) with \( \hat{x}_i \in R \), applying (8) gives

\[
\sum_{j=1}^{i-1} D(n - 2j)P(\hat{x}_{i,j}) + \sum_{j=i+1}^{n} D(n - 2j + 2)P(\hat{x}_{i,j}) = 0.
\]

Both sides of (10) are zero.
On the other hand, if $\widehat{x}_i \notin R$, then $1 < i < n$ and
\begin{equation}
(11) \quad P(\widehat{x}_{i,j}) = 0 \text{ if } |j - i| > 1 \text{ and } P(\widehat{x}_{i-1,i}) = P(\widehat{x}_{i,i+1}).
\end{equation}

The left side of (8) for $x$ can be written, using the definition of $P(\widehat{x}_i)$ and then (6), as
\begin{equation}
= \frac{1}{D(n)} \sum_{1 \leq i \leq n: \widehat{x}_i \in R} D(n - 2i + 1) \left[ \sum_{1 \leq j < i} C(n - 2j)P(\widehat{x}_{i,j}) + \sum_{i < j \leq n} C(n - 2j + 2)P(\widehat{x}_{i,j}) \right]
\end{equation}
\begin{equation}
= \frac{1}{2D(n)} \sum_{1 \leq j < i \leq n: \widehat{x}_i \in R} (D(2n - 2i - 2j + 1) + D(2j - 2i + 1))P(\widehat{x}_{i,j})
\end{equation}
\begin{equation}
+ \frac{1}{2D(n)} \sum_{1 \leq i < j \leq n: \widehat{x}_i \in R} (D(2n - 2i - 2j + 3) + D(2j - 2i - 1))P(\widehat{x}_{i,j}).
\end{equation}

Rearranging, and ignoring the $2D(n)$ in the denominator, gives
\begin{equation}
\sum_{i=1}^{n} 1[\widehat{x}_i \in R] \left[ \sum_{j=1}^{i-1} (D(2n - 2i - 2j + 1) + D(2j - 2i + 1))P(\widehat{x}_{i,j})
\right.
\end{equation}
\begin{equation}
\left. + \sum_{j=i+1}^{n} (D(2n - 2i - 2j + 3) + D(2j - 2i - 1))P(\widehat{x}_{i,j}) \right].
\end{equation}

We must show that (10) and (11) imply that (13) is zero.

We would like to write (13) as a linear combination of expressions that vanish because of (10) and (11) as follows.
\begin{equation}
(14) \quad \sum_{1 \leq i \leq n: \widehat{x}_i \in R} \alpha_i \left[ \sum_{j=1}^{i-1} D(n - 2j)P(\widehat{x}_{i,j}) + \sum_{j=i+1}^{n} D(n - 2j + 2)P(\widehat{x}_{i,j}) \right] + \sum_{1 \leq i \leq n: \widehat{x}_i \in R} \sum_{j=1}^{n} \beta_{i,j}P(\widehat{x}_{i,j}),
\end{equation}
where $\beta_{i,i-1} = \beta_{i,i+1} = 0$. If $1 \leq i < j \leq n$, the coefficient of $P(\widehat{x}_{i,j})$ in (13) is
\begin{equation}
1[\widehat{x}_j \in R] \left[ D(2n - 2i - 2j + 1) + D(2i - 2j + 1) \right]
\end{equation}
\begin{equation}
+ 1[\widehat{x}_i \in R] \left[ D(2n - 2i - 2j + 3) + D(2j - 2i - 1) \right].
\end{equation}

The coefficient of $P(\widehat{x}_{i,j})$ in (14) is
\begin{equation}
(15) \quad 1[\widehat{x}_j \in R] \alpha_j D(n - 2i) + 1[\widehat{x}_i \in R] \alpha_i D(n - 2j + 2) + 1[\widehat{x}_i \notin R] \beta_{i,j} + 1[\widehat{x}_j \notin R] \beta_{j,i}.
\end{equation}

We need to choose the $\alpha$’s and $\beta$’s so that (15) and (16) agree. If $\widehat{x}_i, \widehat{x}_j \in R$, this says
\begin{equation}
D(2n - 2i - 2j + 1) + D(2n - 2i - 2j + 3) = \alpha_j D(n - 2i) + \alpha_i D(n - 2j + 2)
\end{equation}
since $D$ is an odd function. It may sound unreasonable to expect to solve this system, since there are $n$ unknowns and \( \binom{n}{2} \) equations. However, $D$ satisfies relations that make this possible. Solving the equations for small $n$ suggests trying $\alpha_i = 2C(n - 2i + 1)$. The fact that this choice solves these equations for all choices of $n, i, j$ then follows from (6)
and the fact that $D$ is odd. If $\hat{x}_i \notin \mathcal{R}$ and $\hat{x}_j \notin \mathcal{R}$, (15) and (16) agree if $\beta_{i,j} + \beta_{j,i} = 0$. If $\hat{x}_i \in \mathcal{R}$ and $\hat{x}_j \notin \mathcal{R}$, they agree if
\[
D(2n - 2i - 2j + 3) + D(2j - 2i - 1) = \alpha_i D(n - 2j + 2) + \beta_{j,i}.
\]
Using (6) again gives $\beta_{j,i} = 2D(2j - 2i - 1)$. Similarly, if $\hat{x}_i \notin \mathcal{R}$ and $\hat{x}_j \in \mathcal{R}$, they agree if $\beta_{i,j} = 2D(2i - 2j + 1)$. With these choices, $\beta$ is anti-symmetric, and $\beta_{k,k-1} = 2D(1)$ and $\beta_{k,k+1} = 2D(-1)$, so $\beta_{k,k-1} + \beta_{k,k+1} = 0$ as required. This completes the induction argument.

For (9), consider the case of $Q$ first. Use the definition of $P$ to write the right side of (9) as
\[
\frac{C(n+1)}{D(n+1)} \sum_{i=1}^n C(n - 2i + 1) P(\hat{x}_i).
\]
Using (4), this becomes
\[
\frac{1}{2D(n+1)} \sum_{i=1}^n C(2n - 2i + 2) P(\hat{x}_i) + \frac{1}{2} Q(x).
\]
Therefore, we need to prove that
\[
\sum_{i=1}^n \left[ C(2n - 2i + 2) - C(2i) \right] P(\hat{x}_i) = 0.
\]
But by (5), this follows from (8). The proof for $Q^*$ is similar. 

4. PROOF OF THE MAIN RESULT

We will often write $x_1 x_2 \cdots x_n$ instead of $(x_1, x_2, \ldots, x_n)$ below. If $x \in [q]^m$ and $y \in [q]^n$, let $xy$ denote the word $x_1 \cdots x_m y_1 \cdots y_n \in [q]^{m+n}$.

Proof of Theorem 1. We first need to show that the finite dimensional distributions defined in (2) are consistent, i.e., that
\[
\sum_{a \in [q]} P(xa) = P(x), \quad x \in [q]^n, \ n \geq 0.
\]
This is true if $x$ is not proper, since then $xa$ is also not proper, and so both sides vanish. For proper $x$, the proof is by induction on $n$. Note that for $a \in [q],
\[
P(a) = \frac{C(0)}{D(2)} = \frac{1}{q},
\]
so $\sum_{a \in [q]} P(a) = 1$. This gives (17) for $n = 0$. Suppose it holds for all $x \in [q]^{n-1}$ with $n \geq 1$. Then for proper $x \in [q]^n$, using the induction hypothesis in the second equality,
\[
\sum_{a \in [q]} P(xa) = \sum_{a \neq x_n} \frac{1}{D(n+2)} \left[ \sum_{i=1}^n C(n - 2i + 2) P(\hat{x}_i a) + C(-n) P(x) \right] = \frac{1}{D(n+2)} \left[ \sum_{i=1}^n C(n - 2i + 2) P(\hat{x}_i) - C(-n+2) P(x) + (q-1)C(-n) P(x) \right].
\]
The middle term in the second line accounts for the missing term $a = x_n$ when the inductive hypothesis is applied to the case $i = n$ (since $\widehat{x}_n x_n = x$). Using $(j, k, \ell) = (1, n - 2i + 1, 2i)$ in (7) gives
\[
\frac{C(n - 2i + 2)}{D(n + 2)} = \frac{C(n - 2i + 1)}{D(n + 1)} - \frac{C(2i)D(1)}{D(n + 2)D(n + 1)}.
\]
Therefore
\[
\sum_{a \in [q]} P(xa) = P(x) - \frac{Q(x)}{D(n + 2)} - \frac{C(n - 2)}{D(n + 2)} P(x) + (q - 1) \frac{C(n)}{D(n + 2)} P(x).
\]
This is $P(x)$, as required, by (9) and the fact that
\[(q - 1)C(n) = C(n - 2) + C(n + 1)D(1),\]
which is obtained by taking $(j, k, \ell) = (2, -n, n + 1)$ in (7), and then canceling a factor of $\sqrt{q}$.

Invariance of the measure under permutations of colors and translations is immediate from the definition. Invariance under reflection amounts to checking $P(x) = P(x_\cdots x_1)$, which follows from the fact that the coefficients of $\widehat{x}_i$ and $\widehat{x}_{n-i+1}$ in (2), which are $C(n - 2i + 1)$ and $C(-n + 2i - 1)$ respectively, are equal by the symmetry of $C$.

For $1-$dependence, we need to show that for $x \in [q]^m$ and $y \in [q]^n$ with $m, n \geq 0$,
\[P(x \ast y) = P(x)P(y),\]
where the $\ast$ means that there is no constraint at the single site between $x$ and $y$. This is again true if $x$ or $y$ is not proper since then both sides are zero. For proper $x$ and $y$, the proof is by induction, but now on $m + n$. The statement is immediate if $m = 0$ or $n = 0$. So, we take $m \geq 1$ and $n \geq 1$.

There are two cases, according to whether or not $xy$ is a proper coloring, i.e., whether $x_m$ and $y_1$ are equal or different. Assume first that $x_m = y_1$. Without loss of generality, take their common value to be 1. Then using the definition of $P$, including the fact that $P(xy) = 0$,
\[
P(x \ast y) = \sum_{a \in [q]} P(xay) = \frac{1}{D(n + m + 2)} \left[ \sum_{i=1}^m C(n + m - 2i + 2)P(\widehat{x}_i ay) \right.

+ \left. C(n - m)P(xy) + \sum_{j=1}^n C(n - m - 2j)P(xa\widehat{y}_j) \right]

= \frac{1}{D(n + m + 2)} \left[ \sum_{i=1}^m C(n + m - 2i + 2)P(\widehat{x}_i \ast y) + \sum_{j=1}^n C(n - m - 2j)P(x \ast \widehat{y}_j) \right].
\]
Using the induction hypothesis, this becomes
\[
P(x \ast y) = \frac{1}{D(n + m + 2)} \left[ P(y) \sum_{i=1}^m C(n + m - 2i + 2)P(\widehat{x}_i) + P(x) \sum_{j=1}^n C(n - m - 2j)P(\widehat{y}_j) \right].
\]
Taking \((j, k, l) = (n, m - 2i + 1, i)\) in (7) gives
\[
\frac{C(n + m - 2i + 2)}{D(n + m + 2)} = \frac{C(m - 2i + 1)}{D(m + 1)} - \frac{C(2i)D(n + 1)}{D(m + 1)D(n + m + 2)}.
\]
Similarly,
\[
\frac{C(m + 2j - n)}{D(n + m + 2)} = \frac{C(2j - n - 1)}{D(n + 1)} - \frac{C(2n - 2j + 2)D(m + 1)}{D(n + 1)D(n + m + 2)}.
\]
Therefore, since \(C(\cdot)\) is even,
\[
P(x * y) = P(y) \left[ P(x) - \frac{D(n + 1)}{D(n + m + 2)} Q(x) \right] + P(x) \left[ P(y) - \frac{D(m + 1)}{D(n + m + 2)} Q^*(y) \right].
\]
By (9),
\[
P(x * y) = P(x)P(y) \left[ 2 - \frac{C(m + 1)D(n + 1) + C(n + 1)D(m + 1)}{D(n + m + 2)} \right].
\]
Taking \((j, k, l) = (n, m - 2i + 1, i)\) in (7), we see that the expression in brackets above is 1, as required.

Assume now that \(x_m \neq y_1\), say \(x_m = 1\) and \(y_1 = 2\). Then
\[
P(x * y) = \sum_{a \in [q]} P(xay) = \frac{1}{D(n + m + 2)} \sum_{a \neq 1, 2} \left[ \sum_{i=1}^m C(n + m - 2i + 2)P(\hat{x}_i ay) + C(n - m)P(xy) + \sum_{j=1}^n C(n - m - 2j)P(xa\hat{y}_j) \right]
\]
\[
\frac{1}{D(n + m + 2)} \left[ \sum_{i=1}^m C(n + m - 2i + 2)P(\hat{x}_i * y) + \sum_{j=1}^n C(n - m - 2j)P(x * \hat{y}_j) \right]
\]
as in the previous case. However, in the previous case, the term \(P(xy)\) dropped out because \(xy\) was not a proper coloring. In this case, the term \((q - 2)C(n - m)P(xy)\) is cancelled by the terms \(-P(xy)C(n - m + 2)\) and \(-P(xy)C(n - m - 2)\), which arise from
\[
\sum_{a \neq 1, 2} P(\hat{x}_m ay) = P(\hat{x}_m * y) - P(xy) \text{ and } \sum_{a \neq 1, 2} P(xa\hat{y}_1) = P(x * \hat{y}_1) - P(xy).
\]
The fact that the overall coefficient of \(P(xy)\) vanishes is a consequence (4) with \(m = 2\), since \(2C(2) = q - 2\). The rest of the proof is the same as in the case \(x_m = y_1\) above. □

References

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