Poisson Splitting by Factors

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Abstract

Given a homogeneous Poisson process on $\mathbb{R}^d$ with intensity λ, we prove that it is possible to partition the points into two sets, as a deterministic function of the process, and in an isometry-equivariant way, so that each set of points forms a homogeneous Poisson process, with any given pair of intensities summing to λ. In particular, this answers a question of Ball [3], who proved that in $d = 1$, the Poisson points may be similarly partitioned (via a translation-equivariant function) so that *one* set forms a Poisson process of lower intensity, and asked whether the same was possible for all d. We do not know whether it is possible similarly to *add* points (again chosen as a deterministic function of a Poisson process) to obtain a Poisson process of higher intensity, but we prove that this is not possible under an additional finitariness condition.

1 Introduction

Let $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ be the Borel σ-field on $\mathbb{R}^d$. Let $\mathbb{M}$ be the space of all Borel simple point measures on $(\mathbb{R}^d, \mathcal{B})$, and let $\mathcal{M}$ be the product σ-field on $\mathbb{M}$ (we give detailed definitions in Section 2). Given an isometry θ of $\mathbb{R}^d$ and $\mu \in \mathbb{M}$, we define $\theta(\mu)$ to be the measure given by $\theta(\mu)(A) = \mu(\theta^{-1}(A))$ for all $A \in \mathcal{B}$. We say that a measurable mapping $\phi : \mathbb{M} \to \mathbb{M}$ is ***isometry-equivariant*** if $\theta(\phi(\mu)) = \phi(\theta(\mu))$ for all $\mu \in \mathbb{M}$ and for all isometries θ of $\mathbb{R}^d$. Similarly we say that ϕ is ***translation-equivariant*** if it commutes with all translations of $\mathbb{R}^d$. We define a partial order $\leq$ on $\mathbb{M}$ via: $\mu_1 \leq \mu_2$

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if and only if $\mu_1(A) \leq \mu_2(A)$ for all $A \in \mathcal{B}$. We say that a mapping ϕ is ***monotone*** if either $\phi(\mu) \leq \mu$ for all $\mu \in \mathbb{M}$, or $\mu \leq \phi(\mu)$ for all $\mu \in \mathbb{M}$.

Our main result is the following.

Theorem 1. *For all $d \geq 1$ and for all $\lambda > \lambda' > 0$, there exists a monotone isometry-equivariant mapping $\phi : \mathbb{M} \to \mathbb{M}$ such that if X is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ, then $\phi(X)$ and $X - \phi(X)$ are homogeneous Poisson point processes on $\mathbb{R}^d$ with intensities λ' and $\lambda - \lambda'$ respectively.*

In other words, Theorem 1 states that the points of a Poisson process may be coloured red and blue, in a deterministic isometry-equivariant way, so that both the red process and the blue process are Poisson processes. Ball [3] proved that in the case $d = 1$, for all $\lambda > \lambda' > 0$, there exists a monotone translation-equivariant mapping $\phi : \mathbb{M} \to \mathbb{M}$ such that if X is a Poisson point process with intensity λ, then $\phi(X)$ is a homogeneous Poisson point process with intensity λ' (in other words, the Poisson process may be "thinned" in a deterministic translation-equivariant way). Ball asked whether the same was possible in higher dimensions, and also whether the condition of translation-equivariance could be strengthened to isometry-equivariance. Theorem 1 answers both questions affirmatively, as well as providing the additional property that $X - \phi(X)$ is a Poisson process. Evans [4] recently proved that Poisson processes cannot be thinned in an equivariant way with respect to any affine measure-preserving group that is strictly larger than the isometry group.

If all considerations of monotonicity are dropped, then the following result of Ornstein and Weiss applies, even without the restriction that $\lambda > \lambda'$.

Theorem 2 (Ornstein and Weiss)**.** *For all $d \geq 1$ and all $\lambda, \lambda' \in (0, \infty)$, there exists a isometry-equivariant mapping $\phi : \mathbb{M} \to \mathbb{M}$ such that if X is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ, then $\phi(X)$ is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ'.*

Ornstein and Weiss [18] proved Theorem 2 as part of a much more general theory. In particular, they proved the existence of an isomorphism, whereas Theorem 2 asserts the existence only of a homomorphism. The tools we develop to prove Theorem 1 allow us to give an alternative proof of Theorem 2. The map we construct is explicit, and satisfies an additional continuity property (see Theorem 4 below). When $\lambda' > \lambda$, we do not know whether

the condition of monotonicity can be added to Theorem 2 (in other words, whether a Poisson process can be deterministically "thickened").

Question 1. *Let $d \geq 1$ and let $\lambda' > \lambda > 0$. Does there exists a monotone isometry-equivariant $\phi : \mathbb{M} \to \mathbb{M}$ such that if X is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ, then $\phi(X)$ is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ'?*

However, we can prove that the answer to Question 1 becomes *no* when ϕ is required to satisfy the following additional condition. For $\mu \in \mathbb{M}$, we define the restriction of μ to a set $A \in \mathcal{B}$ via: $\mu\big|_A(\cdot) := \mu(\cdot \cap A)$ (so $\mu\big|_A \in \mathbb{M}$). Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^d$. The open ***ball*** of radius r centered at x is denoted by $B(x,r) := \{y : \|x-y\| < r\}$. Let X be a Poisson point process on $\mathbb{R}^d$ with law P. We say that a translation-equivariant measurable mapping $\phi : \mathbb{M} \to \mathbb{M}$ is ***strongly finitary*** with respect to P if, for P-a.e. $\mu \in \mathbb{M}$, there exists a positive real number $n = n(\mu)$ such that for P-a.e. $\mu' \in \mathbb{M}$, we have $\phi(\mu)|_{B(\mathbf{0},1)} = \phi(\mu')|_{B(\mathbf{0},1)}$ whenever $\mu|_{B(\mathbf{0},n)} = \mu'|_{B(\mathbf{0},n)}$. (In other words, the restriction of $\phi(\mu)$ to the unit ball is determined by the restriction of μ to a larger ball, of random but finite radius.) With the addition of this condition, we can answer Question 1 in the negative, even if we drop the condition of isometry-equivariance.

Theorem 3. *Let $d \geq 1$ and $\lambda' > \lambda > 0$. Let X be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ and law P. There does not exist a translation-equivariant monotone measurable mapping $\phi : \mathbb{M} \to \mathbb{M}$ such that $\phi(X)$ is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ' and ϕ is strongly finitary with respect to P.*

In fact, our proof of Theorem 3 will not use the assumption of translation-equivariance either, so we actually prove the stronger statement that no mapping ϕ satisfying the other conditions can have have the property that the restriction of $\phi(\mu)$ to the unit ball is determined by the restriction of μ to a larger random ball, as defined above.

In Section 11, we shall show that the mappings that we produce to prove Theorems 1 and 2 are strongly finitary. The mapping produced in [3] is also strongly finitary.

Theorem 4. *Theorems 1 and 2 hold even with the further requirement that the isometry-equivariant mapping ϕ be strongly finitary with respect to P, where P is the law of X.*

Sometimes deterministic translation-equivariant maps like the ones of Theorems 1 and 2 are called *factors*. Factors are of basic importance in ergodic theory and continue to play a central role in applications of ergodic theory to combinatorics. The combinatorial and probabilistic aspects of factors themselves have received attention in recent years as well. It turns out that factors are intimately related to Palm theory and shift-coupling. For more information, see [26], [27], [8], and [15]. *Factor graphs* of point processes have also received considerable attention; see [5], [7], and [29]. Following [7], a *factor graph* of a point process X is a graph whose vertices are the points of X and whose edges are obtained as a deterministic translation-equivariant function of X. An important special case of a factor graph is a translation-equivariant matching; see [6] for some striking results on this topic. Finally, we refer the interested readers to [18] for very general results regarding factors of Poisson processes and the well-studied isomorphism problem.

One can ask questions similar to ours about factors in a discrete setting. Translation-equivariant matchings of i.i.d. coin flips on $\mathbb{Z}^d$ are considered in [24] and [28]. Much is known about factors of Bernoulli shifts on $\mathbb{Z}$; for example, see the monograph of Ornstein [17]. In particular, it is a classical result of Sinai [23] that if $B(p)$ and $B(q)$ are Bernoulli shifts on $\{0, 1, \ldots, d-1\}^{\mathbb{Z}}$ (i.e., i.i.d. $\{0, 1, \ldots, d-1\}$-valued sequences with laws p and q) and the entropy of p is strictly greater than the entropy of q, then there is a factor from $B(p)$ to $B(q)$. Recently, Ball [2] proved that, if the entropy of p is strictly greater than the entropy of q, and p stochastically dominates q, then in the special case $d = 2$, there is a factor map ϕ from $B(p)$ to $B(q)$ that is monotone (i.e., $\phi(x)_i \leq x_i$ for almost all $x \in \{0, 1, \ldots, d-1\}^{\mathbb{Z}}$ and all $i \in \mathbb{Z}$).

The factor map ϕ given in [2] is also *finitary*; that is, ϕ is continuous when $\{0, 1, \ldots, d-1\}^{\mathbb{Z}}$ is endowed with the product topology. Keane and Smorodinsky improved on results of Ornstein by producing explicit finitary factors between Bernoulli shifts. We refer the interested reader to the original papers of Keane and Smorodinsky [12],[13], and the recent survey article on finitary codes by Serafin [22].

Finally, we also mention the forthcoming work of Angel, Holroyd and Soo [1] concerning monotone deterministic functions of Poisson point processes on finite volumes. In particular, if $\lambda > \lambda'$ and X is a Poisson point process of intensity λ on $[0, 1]$, that article provides a necessary and sufficient condition on (λ, λ') for the existence of a monotone deterministic map $\phi : \mathbb{M} \to \mathbb{M}$ such that $\phi(X)$ is a Poisson point process on $[0, 1]$ of intensity λ'.

2 Some remarks about the proofs

We next motivate the proofs of Theorems 1 and 2 via some simple examples of mappings $\phi : \mathbb{M} \to \mathbb{M}$ having some of the required properties. The proof of Theorem 3 is much shorter and is treated in Section 3. Of course, one of the requirements of ϕ is that it be measurable. All the maps we define will clearly be measurable; we provide the formal definition of the σ-field for $\mathbb{M}$ below.

Measurability. The σ-field $\mathcal{M}$ of subsets of $\mathbb{M}$ is defined in the following way. Let $\mathbb{N}$ be the natural numbers (including zero) and $\bar{\mathbb{N}}$ be $\mathbb{N} \cup \{\infty\}$. For $B \in \mathcal{B}$, the projection map $p_B : \mathbb{M} \to \bar{\mathbb{N}}$ is defined by $p_B(\mu) = \mu(B)$, for all $\mu \in \mathbb{M}$. We let $\mathcal{M}$ be the smallest σ-field such that all the projection maps are measurable. $\Diamond$

Note that throughout this paper, the only laws we consider on $\mathcal{M}$ will be homogeneous Poisson point processes on $\mathbb{R}^d$ and their restrictions to subsets of $\mathbb{R}^d$. We say that U is a $\mathrm{U}[0,1]$ random variable if it is uniformly distributed in $[0,1]$. Similarly, we say that V is a $\mathrm{U}[B]$ random variable if it is uniformly distributed in some Borel set B. In the next examples and throughout this paper, we shall assert that certain random variables can be expressed as functions of $\mathrm{U}[0,1]$ random variables. This can be justified by appealing to the Borel isomorphism theorem [25, 3.4.24]. However, very often we need only the following two results, which are consequences of the Borel isomorphism theorem. Because of the need for isometry-equivariance in our constructions, we shall often need to be rather explicit about such functions.

Lemma 5 (Reproduction)**.** *There exist measurable deterministic functions $\{g_i\}_{i\in\mathbb{N}}$, where $g_i : [0,1] \to [0,1]$, such that if U is a $\mathrm{U}[0,1]$ random variable, then $\{g_i(U)\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. $\mathrm{U}[0,1]$ random variables.*

For an explicit proof, see, for example, [11, Lemma 3.21].

Lemma 6 (Coupling)**.** *Let $\lambda > 0$. There exists a collection of measurable mappings $\phi^{\mathrm{P}} = \left\{\phi^{\mathrm{P}}_A\right\}_{A\in\mathcal{B}}$, where for each $A \in \mathcal{B}$, the map $\phi^{\mathrm{P}}_A = \phi^{\mathrm{P}}_{(A,\lambda)} : [0,1] \to \mathbb{M}$ is such that if U is a $\mathrm{U}[0,1]$ random variable, then $\phi^{\mathrm{P}}_A(U)$ is a Poisson point process on A with intensity λ.*

Example 1 (A $\mathbb{Z}^d$-translation-equivariant mapping between Poisson point processes of arbitrary intensities)**.** Let $\lambda, \lambda' > 0$. Let X be a Poisson point

process on $\mathbb{R}^d$ with intensity λ and law P. Let $C_{\mathbf{0}}$ be a cube of side-length 1 containing the origin $\mathbf{0} \in \mathbb{Z}^d$, and let $C_i := C_{\mathbf{0}} + i$ for $i \in \mathbb{Z}^d$. Assume that $C_{\mathbf{0}}$ is such that the collection $\mathcal{P} = \{C_i\}_{i \in \mathbb{Z}^d}$ is a partition of $\mathbb{R}^d$. The mapping ϕ will be defined by specifying $\phi(\cdot)\big|_C$ for all $C \in \mathcal{P}$. We shall define ϕ only off a P-null set; it is not difficult to extend ϕ to all of $\mathbb{M}$ so that it still commutes with all translations of $\mathbb{Z}^d$. Let $g : [0,1] \to \mathbb{M}$ be a measurable function such that if U is a $\mathrm{U}[0,1]$ random variable, then $g(U)$ is a Poisson process on $C_{\mathbf{0}}$ with intensity λ'. We shall define a measurable map $h : \mathbb{M} \to [0,1]^{\mathbb{Z}^d}$ with the following properties: $h(X)$ is a collection of i.i.d. $\mathrm{U}[0,1]$ random variables, and for all translations θ of $\mathbb{Z}^d$ we have $h(\theta(X))_i = h(X)_{\theta(i)}$ for all $i \in \mathbb{Z}^d$. For all $i \in \mathbb{Z}^d$, let $\theta_i(x) = x + i$ for all $x \in \mathbb{R}^d$. Given the mapping h, it easy to see that by taking

$$\phi(X)\big|_{C_i} := \theta_{-i}(g(h(X)_i))$$

for all $i \in \mathbb{Z}^d$, we have that ϕ commutes with translations of $\mathbb{Z}^d$ and that $\phi(X)$ is a Poisson point process on $\mathbb{R}^d$ with intensity λ'. It remains to define h.

If $X(C) = 1$, then we say that C is ***special***. Let $K(i)$ be the index of the first special cube to the right of cube i; that is, $K(i) = i + (n, 0, \ldots, 0)$ where $n = n(i)$ is the smallest non-negative integer such that $C_{i+(n,0,\ldots,0)}$ is special. Note that P-a.s. K is well defined. For each special cube C_i, let $z(i)$ be the unique point $x \in C_i$ such that $X(\{x\}) = 1$. Since X is a Poisson point process, the random variables $\{X|_{C_i}\}_{i \in \mathbb{Z}^d}$ are independent, and also conditional on the event that C_i is special, $z(i)$ is a $\mathrm{U}[C_i]$ random variable. Let $f : C_{\mathbf{0}} \to [0,1]^{\mathbb{N}}$ be a measurable function such that if V is a $\mathrm{U}[C_{\mathbf{0}}]$ random variable, then $f(V)$ is a sequence of i.i.d. $\mathrm{U}[0,1]$ random variables. For all $i \in \mathbb{Z}^d$, let

$$h(X)_i := f\big(z(K(i)) - K(i)\big)_{n(i)}\,.$$

It is easy to verify that h satisfies the required properties. $\Diamond$

Let us remark that in Example 1 we used the fact that if X is a Poisson process, then conditional that it has one point in A, the location of that point is a $\mathrm{U}[A]$ random variable. This elementary fact will often be useful and we shall appeal to it again in the next example and in the proofs of Theorems 1 and 2. We refer the reader to [21, Theorem 1.2.1] or [14] for background and state a slightly more general result in the lemma below. Let $\mathcal{L}$ denote Lebesgue measure.

Lemma 7. *Let X be a Poisson point process on $\mathbb{R}^d$ with intensity λ. Let $A \in \mathcal{B}$ be a Borel set with finite Lebesgue measure. Let K be a Poisson random variable with mean $\lambda\mathcal{L}(A)$. Let $\{V_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d.* $\mathrm{U}[A]$ *random variables that are independent of K. Then $X\big|_A$ has the same law as $Z := \sum_{i=1}^{K} \delta_{V_i}$.*

A central requirement in Theorems 1, 2, and 3 is that ϕ be a deterministic function of X. The mapping in Example 1 is a deterministic function of X and commutes with all translations of $\mathbb{Z}^d$. Given a $\mathrm{U}[C_{\mathbf{0}}]$ random variable V, independent of X, we can modify Example 1 by starting with a randomly shifted partition $\{C_i + V\}_{i\in\mathbb{Z}^d}$ of $\mathbb{R}^d$ and obtain a mapping Φ that is a function of X and V. As a result of starting with a randomly shifted partition, the joint distribution of $(X, \Phi(X, V))$ is fully translation-invariant. However, Φ is no longer a deterministic function of X.

Instead of using the lattice $\mathbb{Z}^d$, we shall use randomness from the process to define a partition of $\mathbb{R}^d$. It is straightforward to do this in an isometry-equivariant way. The difficulty lies in choosing a partition that avoids potential dependency problems.

We now turn our attention to Theorem 1. Let $\lambda > \lambda' > 0$. It is non-trivial to show that there exists a (not necessarily translation-equivariant) monotone mapping which maps a Poisson point process of intensity λ to a Poisson point process of intensity λ'. In Example 1, we asserted the existence of a certain coupling between uniform random variables and Poisson point processes via a measurable function $g : [0, 1] \to \mathbb{M}$ such that whenever U is a $\mathrm{U}[0, 1]$ random variable, $g(U)$ is a Poisson point process. Due to the monotonicity requirement in Theorem 1, we require a more specialized coupling.

An important tool in the proof of Theorem 1 will be Proposition 8 below, which is motivated by one of the key ideas from [3, Lemma 3.1]. Proposition 8 provides a coupling between a Poisson point process X in a finite volume and another, Y, of lower intensity, such that $Y \leq X$ and the process $X - Y$ is also a Poisson point process. The process Y is not a deterministic function of X, but the coupling has certain other useful properties.

Throughout this paper, it will be convenient to encode randomness as a function of $\mathrm{U}[0, 1]$ random variables, as was done repeatedly in Example 1. For any point process Z, the ***support*** of Z is the random set

$$[Z] := \left\{x \in \mathbb{R}^d : Z(x) = 1\right\} .$$

Elements of $[Z]$ are called Z***-points***. We call a mapping $\Phi : \mathbb{M} \times [0, 1] \to \mathbb{M}$

a ***splitting*** if $\Phi(\mu, u) \leq \mu$ for all $(\mu, u) \in \mathbb{M} \times [0,1]$ and if both $\Phi(X, U)$ and $X - \Phi(X, U)$ are Poisson point processes whenever X is a Poisson point process and U is a $\mathrm{U}[0,1]$ random variable independent of X. For example, consider the coupling between a Poisson point process X on $\mathbb{R}^d$ of intensity λ and another, Y, of lower intensity λ', that is given by colouring the points of X independently of each other red or blue with probabilities $\frac{\lambda'}{\lambda}$ and $1 - \frac{\lambda'}{\lambda}$ and then taking the red points to be the set of Y-points. It is easy to see that both Y (the red points) and $X - Y$ (the blue points) are independent Poisson point processes on $\mathbb{R}^d$ with intensities λ' and $\lambda - \lambda'$. This elementary result is sometimes referred to as the *colouring theorem* [14] and this coupling can be expressed as a splitting since all the required coin-flips can be encoded as a function of a single $\mathrm{U}[0,1]$ random variable. We shall revisit this elementary coupling in more detail in Section 5. The coupling given by Proposition 8 below is also a splitting.

Proposition 8 (Splitting on finite volumes)**.** *Let $\lambda > \lambda' > 0$. There exists a finite constant $K = K(\lambda, \lambda')$ and a family ϕ^{fin} of measurable mappings ϕ^{fin}_A so that for each $A \in \mathcal{B}$ with finite Lebesgue measure larger than K, the map $\phi^{\mathrm{fin}}_A = \phi^{\mathrm{fin}}_{(A,\lambda,\lambda')} : \mathbb{M} \times [0,1] \to \mathbb{M}$ has the following properties:*

(a) The map ϕ^{fin}_A is monotone; that is, $\phi^{\mathrm{fin}}_A(\mu, u) \leq \mu$ for all $(\mu, u) \in \mathbb{M} \times [0,1]$.

(b) For all $(\mu, u) \in \mathbb{M} \times [0,1]$, we have $\phi^{\mathrm{fin}}_A(\mu, u) = \phi^{\mathrm{fin}}_A(\mu\big|_A, u)$.

(c) If X is homogeneous Poisson point processes on $\mathbb{R}^d$ with intensity λ and U is a $\mathrm{U}[0,1]$ random variable independent of X, then $\phi^{\mathrm{fin}}_A(X\big|_A, U)$ is a Poisson point process of intensity λ' on A, and $X\big|_A - \phi^{\mathrm{fin}}_A(X\big|_A, U)$ is a Poisson point process of intensity $\lambda - \lambda'$ on A.

(d) For all $(\mu, u) \in \mathbb{M} \times [0,1]$, if $\mu(A) = 1$, then $\phi^{\mathrm{fin}}_A(\mu, u) = 0$, while if $\mu(A) = 2$, then $\phi^{\mathrm{fin}}_A(\mu, u) = \mu$.

(e) The family of mappings ϕ^{fin} has the following isometry-equivariance property: for any isometry θ of $\mathbb{R}^d$, and for all $(\mu, u) \in \mathbb{M} \times [0,1]$,

$$\theta\left(\phi^{\mathrm{fin}}_A(\mu, u)\right) = \phi^{\mathrm{fin}}_{\theta(A)}(\theta(\mu), u)\,.$$

We shall prove Proposition 8 in Section 4. Property (d) of Proposition 8 will be vital to the proof of Theorem 1. It states that whenever $X\big|_A$ has exactly one point in its support, $\phi^{\mathrm{fin}}_A(X\big|_A, U)$ will have no points, while

whenever $X\big|_A$ has exactly two points in its support, $X\big|_A - \phi_A^{\text{fin}}(X\big|_A, U)$ will have no points. Hence when $X\big|_A$ has exactly one or two points the locations of these points provide a possible source of randomness. The next example will illustrate how property (d) is exploited and will help to motivate the proof of Theorem 1. To make use of property (d), we shall need the following elementary lemma.

Let $\oplus$ denote addition modulo one; that is, for $x, y \in \mathbb{R}$, let $x \oplus y$ be the unique $z \in [0, 1)$ such that $x + y - z \in \mathbb{Z}$.

Lemma 9 (Adding U[0, 1] random variables modulo 1)**.** *Let U_1 and U_2 be* U[0, 1] *random variables that are measurable with respect to the σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ and such that U_1 is independent of $\mathcal{F}_2$ and U_2 is independent of $\mathcal{F}_1$. If $U := U_1 \oplus U_2$, then U is independent of $\mathcal{F}_1$, U is independent of $\mathcal{F}_2$, and U is a* U[0, 1] *random variable.*

Proof. The proof follows from the Fubini theorem and the fact that for every $x \in \mathbb{R}$ we have $U_1 \oplus x \stackrel{d}{=} U_1$. Let $E \in \mathcal{F}_2$ and let Q be the joint law of U_2 and $\mathbf{1}_E$. Let $B \in \mathcal{B}$. By symmetry, it is enough to show that $\mathbb{P}(\{U \in B\} \cap E) = \mathbb{P}(U_1 \in B)\mathbb{P}(E)$. By the independence of U_1 and $\mathcal{F}_2$, we have

$$\begin{aligned}\mathbb{P}(\{U \in B\} \cap E) &= \int \mathbb{P}(U_1 \oplus x \in B) i \, dQ(x, i) \\ &= \int \mathbb{P}(U_1 \in B) i \, dQ(x, i) \\ &= \mathbb{P}(U_1 \in B)\mathbb{P}(E) \,. \end{aligned}$$

□

Example 2 (A monotone map $\phi : \mathbb{M} \to \mathbb{M}$ which maps a Poisson process X to another of lower intensity such that $X - \phi(X)$ is also a Poisson process)**.** Let $\lambda > \lambda' > 0$. Let X be a Poisson point process on $\mathbb{R}^d$ with intensity λ and law P. Let $\mathcal{P} = \{C_i\}_{i \in \mathbb{N}}$ be an indexed partition of $\mathbb{R}^d$ into equal sized cubes, all translates of one another, large enough so that the Lebesgue measure of each cube is larger than the constant K from Proposition 8. The monotone mapping ϕ will be defined by specifying $\phi(\cdot)\big|_C$ for all $C \in \mathcal{P}$.

Let $U = \{U_i\}_{i \in \mathbb{N}}$ be a sequence of i.i.d. U[0, 1] random variables that are independent of X. Let

$$\Phi(X, U) := \sum_{i \in \mathbb{N}} \phi_{C_i}^{\text{fin}}(X, U_i) \,,$$

where ϕ^{fin} is the splitting from Proposition 8. The map ϕ will be defined so that $\phi(X) \stackrel{d}{=} \Phi(X,U)$ and $X - \phi(X) \stackrel{d}{=} X - \Phi(X,U)$. By properties (c) and (b) of Proposition 8, we deduce that $\phi(X)$ and $X - \phi(X)$ are Poisson point process on $\mathbb{R}^d$ with intensities λ' and $\lambda - \lambda'$.

If $X(C) = 1$, then we say that C is ***one-special***, while if $X(C) = 2$, then we say that C is ***two-special***. Let k^1 and k^2 be the indices of the one-special and two-special cubes with the least index, respectively. Note that P-a.s. k^1 and k^2 are well defined. Let Z^1 be the unique X-point in C_{k^1}. Let Z^2_1 and Z^2_2 be the two X-points in C_{k^2}, where Z^2_1 is the one closest to the origin. Let $C_{\mathbf{0}}$ be the cube containing the origin. Fix a measurable function $f_{C_{\mathbf{0}}} : C_{\mathbf{0}} \to [0,1]$ such that if V is a $\mathrm{U}[C_{\mathbf{0}}]$ random variable, then $f_{C_{\mathbf{0}}}(V)$ is a $\mathrm{U}[0,1]$ random variable. For each $C \in \mathcal{P}$, let $c \in C$ be so that $C - c = C_{\mathbf{0}}$ and let $f_C : C \to [0,1]$ be defined via $f_C(x) = f_{C_{\mathbf{0}}}(x - c)$. Since X is a Poisson point process, it follows from Lemma 7 that conditional on k^1 we have that Z^1 is a $\mathrm{U}[C_{k^1}]$ random variable. Moreover it is easy to see that $S^1 := f_{C_{k^1}}(Z^1)$ is in fact a $\mathrm{U}[0,1]$ random variable independent of

$$\mathcal{F}^1 := \sigma\big(\mathbf{1}_{[X(C_i)\neq 1]} X\big|_{C_i} : i \in \mathbb{N}\big)\,.$$

Similarly, it is easy to define S^2 as a function of (Z^2_1, Z^2_2) so that S^2 is a $\mathrm{U}[0,1]$ random variable independent of

$$\mathcal{F}^2 := \sigma\big(\mathbf{1}_{[X(C_i)\neq 2]} X\big|_{C_i} : i \in \mathbb{N}\big)\,,$$

namely,

$$S^2 := f_{C_{k^2}}(Z^2_1) \oplus f_{C_{k^2}}(Z^2_2)\,.$$

To see why the above definition works, consider the random variables Y_1 and Y_2 defined as follows. Choose with a toss of a fair coin (that is independent of X) one of Z^2_1 or Z^2_2 to be Y_1 and let Y_2 be so that $\{Y_1, Y_2\} = \{Z^2_1, Z^2_2\}$. Clearly Y_1 and Y_2 are independent $\mathrm{U}[C_{k^2}]$ random variables and $S^2 = f_{C_{k^2}}(Y_1) \oplus f_{C_{k^2}}(Y_2)$.

Note that S^1 is measurable with respect to $\mathcal{F}_2$ and S^2 is measurable with respect to $\mathcal{F}^1$. Let

$$S := S^1 \oplus S^2\,.$$

By Lemma 9, we have that S is independent of $\mathcal{F}_1$ and S is independent of $\mathcal{F}_2$. For all $i \in \mathbb{N}$, let

$$\phi(X)\big|_{C_i} := \phi^{\text{fin}}_{C_i}\big(X, g_i(S)\big)\,,$$

where g_i is the sequence of functions from Lemma 5. By property (b) of Proposition 8, we see that ϕ is monotone. We shall now show that $\phi(X) \stackrel{d}{=} \Phi(X,U)$ and $X - \phi(X) \stackrel{d}{=} X - \Phi(X,U)$.

Observe that by property (d) of Proposition 8, for each one-special cube C we have

$$\phi(X)\big|_C = \Phi(X,U)\big|_C = 0\,.$$

Since S is independent of $\mathcal{F}^1$ and $\{g_i(S)\}_{i\in\mathbb{N}} \stackrel{d}{=} \{U_i\}_{i\in\mathbb{N}}$, we have that

$$\sum_{i\in\mathbb{N}} \mathbf{1}_{[X(C_i)\neq 1]}\phi^{\text{fin}}_{C_i}(X, g_i(S)) \stackrel{d}{=} \sum_{i\in\mathbb{N}} \mathbf{1}_{[X(C_i)\neq 1]}\phi^{\text{fin}}_{C_i}(X, U_i)\,.$$

Thus $\phi(X) \stackrel{d}{=} \Phi(X,U)$. Similarly, by property (d) of Proposition 8, for each for each one-special cube C we have

$$(X - \phi(X))\big|_C = (X - \Phi(X,U))\big|_C = 0\,.$$

Since S is independent of $\mathcal{F}^2$, we have that

$$\sum_{i\in\mathbb{N}} \mathbf{1}_{[X(C_i)\neq 2]}\big(X\big|_{C_i} - \phi^{\text{fin}}_{C_i}(X, g_i(S))\big) \stackrel{d}{=} \sum_{i\in\mathbb{N}} \mathbf{1}_{[X(C_i)\neq 2]}\big(X\big|_{C_i} - \phi^{\text{fin}}_{C_i}(X, U_i)\big)\,.$$

Thus $X - \phi(X) \stackrel{d}{=} X - \Phi(X,U)$. $\Diamond$

As an aside, one might ask whether the two Poisson processes X and $X - \phi(X)$ in Example 2 or Theorem 1 can be made independent of each other, but it turns out that this is easily ruled out. (It may come as a surprise that two dependent Poisson processes can have a sum that is still a Poisson process; see [10].)

Proposition 10. *There does not exist a monotone map $\phi : \mathbb{M} \to \mathbb{M}$ such that if X is a homogeneous Poisson point process on $\mathbb{R}^d$, then $\phi(X)$ and $X - \phi(X)$ are independent homogeneous Poisson point processes on $\mathbb{R}^d$ with strictly positive intensities.*

Proof. Let X be a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$. Let $\alpha \in (0,1)$. Towards a contradiction assume that $\phi(X)$ and $X - \phi(X)$ are independent Poisson point processes in $\mathbb{R}^d$ with intensities $\alpha\lambda$ and $(1-\alpha)\lambda$.

Let $\mathfrak{R}$ and $\mathfrak{B}$ be independent Poisson point processes on $\mathbb{R}^d$ with intensities $\alpha\lambda$ and $(1-\alpha)\lambda$. Note that

$$(\mathfrak{R}, \mathfrak{B}, \mathfrak{R}+\mathfrak{B}) \stackrel{d}{=} (\phi(X), X-\phi(X), X)\,. \tag{1}$$

Now let $Z := \mathfrak{R}+\mathfrak{B}$ and let $B = B(0,1)$ and consider the events

$$E \;:=\; \{Z(B)=1\} \cap \{\mathfrak{R}(B)=1\}$$

and

$$E' \;:=\; \{X(B)=1\} \cap \{\phi(X)(B)=1\}\,.$$

Clearly, $\mathbb{P}(E \mid Z) = \alpha \mathbf{1}_{[Z(B)=1]}$, but since $E' \in \sigma(X)$, we have that $\mathbb{P}(E' \mid X) = \mathbf{1}_{E'}$. Since $\alpha \in (0,1)$, we conclude that $\mathbb{P}(E \mid Z) \stackrel{d}{\neq} \mathbb{P}(E' \mid X)$, which contradicts (1). $\square$

Outline of the proofs. Following the lead of Examples 1 and 2, we shall introduce an isometry-equivariant partition of $\mathbb{R}^d$. The partition will consist of globes, which will be specially chosen balls of a fixed radius, together with a single unbounded part. The partition will be chosen as a deterministic function of the Poisson process by a procedure that does not need to examine the Poisson points inside the globes. The precise definition of this partition and its properties are somewhat subtle; see Sections 5 and 6. The most important property is that conditional on the partition, the process restricted to the bounded parts is a Poisson point process that is independent of the process on the unbounded part. This may be regarded as an extension of the following property enjoyed by stopping times for a one-dimensional Poisson process: conditional on the stopping time, the process in the future is a Poisson process independent of the process in the past. The precise formulation of the property we need may be found in Proposition 16.

To prove Theorem 1, we shall employ the splitting from Proposition 8 on the bounded parts as in Example 2. The Poisson points in the unbounded part will be split independently of each other with probabilities $(\frac{\lambda'}{\lambda}, 1-\frac{\lambda'}{\lambda})$. When one of the balls of the partition contains exactly one or two points, the splitting from Proposition 8 is completely deterministic. Thus the locations of these points provides a source of randomness that can be used to facilitate the splitting from Proposition 8 on the other balls of the partition, as in Example 2, and in addition can be used to independently split the points that do not belong to a bounded part. Of course, we cannot use randomness

precisely as in Example 2 since that privileges the origin and therefore is not equivariant. Instead, we use randomness from the available source that is (essentially) nearest to where it is used.

Aside from some careful bookkeeping to ensure isometry-equivariance, the two main ingredients for the proof of Theorem 1 are: an isometry-equivariant partition with the independence property described above, and the splitting from Proposition 8. Next we focus our discussion on these two ingredients.

The radius R of the balls of the isometry-equivariant partition will depend on (λ, λ', d). For all $x \in \mathbb{R}^d$ and all $0 < s < r$, we define the ***shell*** centered at x from s to r to be the set

$$A(x; s, r) := \left\{ y \in \mathbb{R}^d : s \leq \|x - y\| \leq r \right\} .$$

Let X be a Poisson point process on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. A single ball of radius R contained in $B(x, R + 80)$ will be chosen to be a globe (a member of the partition) only if two properties are satisfied: the shell $A(x; R + 90 + d, R + 100 + d)$ contains no X-points; and the shell $A(x; R + 80, R + 90 + d)$ is relatively densely filled with X-points, that is, every ball of radius $1/2$ that is contained in $A(x; R + 80, R + 90 + d)$ itself contains an X-point. A minor complication is that the set of $x \in \mathbb{R}^d$ satisfying these properties is not discrete, but consists of small well-separated clusters. The centers of mass of these clusters will be the centers of the globes.

The key step in defining the splitting in Proposition 8 is to construct a coupling of Poisson random variables with the analogous properties of Proposition 8 (save isometry-equivariance). We shall obtain the joint mass function of the required coupling by taking finite perturbations of the joint mass function for two independent Poisson random variables. The calculations become straightforward when we view the mass function as an infinite matrix satisfying certain row and column constraints.

The isometry-equivariant partition used in the proof of Theorem 1 is used again in the proof of Theorem 2; given this partition, the ideas in Example 1 can be easily adapted to prove a (weaker) translation-equivariant version of Theorem 2. It requires some additional effort to prove Theorem 2 in its entirety. The proof of Theorem 4 is not difficult and will follow from the definitions of the maps in Theorems 1 and 2.

Organization of the paper. The rest of paper proceeds as follows. In Section 3 we prove Theorem 3. This section is independent of the other

sections. Section 4 is devoted to a proof of Proposition 8. In Sections 5 and 6 we specify the properties that the isometry-equivariant partition must satisfy, and prove that such a partition does indeed exist. In Section 7 we define some desired properties of a procedure that assigns randomness from the globes that contain exactly one or two points to the other globes and to the points of the unbounded part. The proof of Theorem 1 is given in Section 8 and the existence of the procedure that assigns randomness is proved in Section 9. In Section 10 we prove Theorem 2. In Section 11 we prove Theorem 4. Finally, in Section 12 we state some open problems.

3 Proof of Theorem 3

In this section we shall prove Theorem 3. The proof is by contradiction. The basic idea is as follows. Let X be a Poisson point process on $\mathbb{R}^d$ with positive intensity λ and law P. Let $\phi : \mathbb{M} \to \mathbb{M}$ be strongly finitary with respect to P such that $\phi(X)$ is a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda' > \lambda$ and $X \leq \phi(X)$. Since $\phi(X)$ has greater intensity than X, with non-zero probability we have $X(B(\mathbf{0}, 1)) = 0$ and $\phi(X)(B(\mathbf{0}, 1)) \geq 1$. Since ϕ is strongly finitary with respect to P, there is a fixed M such that with non-zero probability, we also have $\phi(X)|_{B(\mathbf{0},1)} = \phi(X')|_{B(\mathbf{0},1)}$, where X' is equal to X on $B(\mathbf{0}, M)$ but is resampled off $B(\mathbf{0}, M)$. Define a new simple point process Z from $\phi(X)$ by deleting all points in $B(\mathbf{0}, 1)$ and by deleting each point in $[\phi(X)|_{B(\mathbf{0},1)^c}]$ independently with probability λ/λ' conditional on $\phi(X)$. See Figure 1 for an illustration. Since $\phi(X)$ is a Poisson point process, $\phi(X)|_{B(\mathbf{0},1)}$ is independent of $\phi(X)|_{B(\mathbf{0},1)^c}$ and we may define Z so that it is independent of $\phi(X)|_{B(\mathbf{0},1)}$. Since $X \leq \phi(X)$, there is a non-zero probability that $Z\big|_{B(\mathbf{0},M)} = X\big|_{A(\mathbf{0};1,M)}$. Moreover, conditional on the event that $X(B(\mathbf{0}, 1)) = 0$ and $\phi(X)(B(\mathbf{0}, 1)) \geq 1$, there is a non-zero probability that $\phi(Z)|_{B(\mathbf{0},1)} = \phi(X)|_{B(\mathbf{0},1)}$. Clearly, this contradicts the independence of Z from $\phi(X)|_{B(\mathbf{0},1)}$; the following lemma formalizes this intuition.

Lemma 11. *Let $(S, \mathcal{S})$ be a measurable space. If X and Y are independent random variables taking values in S and $A := \{y \in S : \mathbb{P}(Y = y) > 0\}$, then $\mathbb{P}(\{X = Y\} \cap \{Y \in A^c\}) = 0$.*

Proof. We apply the Fubini theorem and the independence of X and Y as

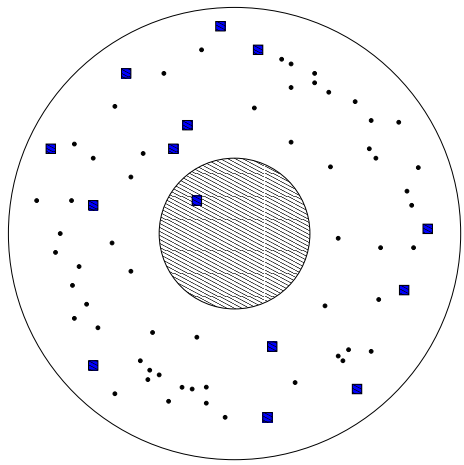

Figure 1: The dots are the original points of X and the squares are points of $\phi(X) \setminus X$. The shaded region is $B(\mathbf{0}, 1)$ and the unshaded ring is $A(\mathbf{0}; 1, M)$. By selecting subsets of the points in $A(\mathbf{0}; 1, M)$ uniformly at random there is non-zero probability that we shall select all the dots.

follows. Let μ_X be the law of X. Then

$$\begin{aligned}\mathbb{P}(\{X = Y\} \cap \{Y \in A^c\}) &= \mathbb{P}(\{X = Y\} \cap \{X \in A^c\}) \\ &= \int_{A^c} \mathbb{P}(Y = x)\, d\mu_X(x) \\ &= \int_{A^c} 0 \; d\mu_X(x) = 0\,. \qquad \square\end{aligned}$$

With Lemma 11 we can now make the above argument for Theorem 3 precise.

Proof of Theorem 3. Let $\lambda' > \lambda > 0$. Towards a contradiction, let X be a Poisson point process on $\mathbb{R}^d$ with intensity λ and law P. Let $\phi : \mathbb{M} \to \mathbb{M}$ be a mapping that is strongly finitary with respect to P such that $X \leq \phi(X)$ and $\phi(X)$ is a Poisson point process on $\mathbb{R}^d$ with intensity λ'. Since $X \leq \phi(X)$ and $\phi(X)$ has greater intensity, we must have that

$$\mathbb{P}\Big(\{\phi(X)(B(\mathbf{0}, 1)) \geq 1\} \cap \{X(B(\mathbf{0}, 1)) = 0\} \Big) > 0\,.$$

For $\mu \in \mathbb{M}$, let $N = N(\mu)$ be the smallest natural number such that for P-a.e. $\mu' \in \mathbb{M}$ we have $\phi(\mu)|_{B(\mathbf{0},1)} = \phi(\mu')|_{B(\mathbf{0},1)}$ whenever $\mu|_{B(\mathbf{0},N)} = \mu'|_{B(\mathbf{0},N)}$. Let

$$E := \{N(X) < M\} \ \cap \ \{\phi(X)(B(\mathbf{0},1)) \geq 1\} \ \cap \ \{X(B(\mathbf{0},1)) = 0\}$$

for some $M > 0$. Since ϕ is strongly finitary with respect to P, we have that $\mathbb{P}\left(N(X) < \infty\right) = 1$ and we may choose M so that

$$\mathbb{P}(E) > 0\,. \tag{2}$$

Note that on the event E we have that

$$\phi(X)|_{B(\mathbf{0},1)} = \phi\big(X|_{A(\mathbf{0};1,M)} + W|_{B(\mathbf{0},M)^c}\big)|_{B(\mathbf{0},1)}\,,$$

where W is independent of X and has law P. Let U be a $\mathrm{U}[0,1]$ random variable independent of X and W. We shall show that there exists a measurable function $H : \mathbb{M} \times \mathbb{M} \times [0,1] \to \mathbb{M}$ such that

$$\mathbb{P}\Big(\ \big\{H\big(\phi(X)|_{A(\mathbf{0};1,M)}, W, U\big) = \phi(X)|_{B(\mathbf{0},1)}\big\} \ \cap \ E\Big) > 0\,. \tag{3}$$

Define a measurable function $s : \mathbb{M} \times [0,1] \to \mathbb{M}$ such that: if $\mu(\mathbb{R}^d) = \infty$, then $s(\mu, u) = 0$ for all $u \in [0,1]$, while if $\mu(\mathbb{R}^d) < \infty$, then $[s(\mu, U)]$ is a uniformly random subset of $[\mu]$. Since $X \leq \phi(X)$ and since U is independent of X, we claim that for any event $E' \in \sigma(X)$ with $\mathbb{P}(E') > 0$,

$$\mathbb{P}\Big(\ \big\{s(\phi(X)|_{A(\mathbf{0};1,M)}, U) = X|_{A(\mathbf{0};1,M)}\big\} \ \cap \ E'\Big) > 0\,. \tag{4}$$

To verify (4), let

$$L := \int_0^1 \mathbf{1}\Big[s(\phi(X)|_{A(\mathbf{0};1,M)}, u) = X|_{A(\mathbf{0};1,M)}\Big]\, du\,.$$

By the Fubini theorem and the independence of X and U, we have that

$$\mathbb{P}\Big(\ \big\{s(\phi(X)|_{A(\mathbf{0};1,M)}, U) = X|_{A(\mathbf{0};1,M)}\big\} \ \cap \ E'\Big) = \mathbb{E}L\mathbf{1}_{E'}\,.$$

Observe that from the definition of s and the fact that $X \leq \phi(X)$, we must have that $L > 0$ P-a.s. Since $\mathbf{1}_{E'} \geq 0$ and $\mathbb{E}\mathbf{1}_{E'} > 0$, it follows that $\mathbb{E}L\mathbf{1}_{E'} > 0$.

Hence taking $E' = E$, from (2) and (4) we have that

$$\mathbb{P}\left(\left\{s(\phi(X)|_{A(\mathbf{0};1,M)}, U) = X|_{A(\mathbf{0};1,M)}\right\} \cap E\right) > 0\,. \tag{5}$$

For all $(\mu, \mu', u) \in \mathbb{M} \times \mathbb{M} \times [0,1]$, define

$$H(\mu, \mu', u) := \phi\big(s(\mu|_{A(\mathbf{0};1,M)}, u) + \mu'|_{B(\mathbf{0},M)^c}\big)|_{B(\mathbf{0},1)}\,.$$

From (5), the definition of H and the definition of E, it is obvious that (3) holds.

Since $\phi(X)$ is a Poisson point process, $\phi(X)|_{B(\mathbf{0},1)}$ and $\phi(X)|_{A(\mathbf{0};1,M)}$ are independent and since U and W are independent of X, we have that $\phi(X)|_{B(\mathbf{0},1)}$ is independent of $H\big(\phi(X)|_{A(\mathbf{0};1,M)}, W, U\big)$. In addition, $\mathbb{P}(\phi(X)|_{B(\mathbf{0},1)} = \mu) = 0$ for all $\mu \in \mathbb{M} \setminus \{0\}$ and $\phi(X)|_{B(\mathbf{0},1)} \neq 0$ on the event E. Thus equation (3) contradicts Lemma 11. □

4 Proof of Proposition 8

The proof of Proposition 8 is based on a specific coupling of two Poisson random variables.

Lemma 12. *For any $\alpha \in (0,1)$, there exists a $k(\alpha)$ such that if $\lambda > k(\alpha)$, then there exist random variables X and Y such that X, Y and $X+Y$ have Poisson distributions with respective means $\alpha\lambda$, $(1-\alpha)\lambda$ and λ, and*

$$\mathbb{P}(Y = 0 \mid X + Y = 1) = 1 = \mathbb{P}(X = 0 \mid X + Y = 2)\,.$$

Proof. Write $\pi_i^\gamma := e^{-\gamma}\gamma^i/i!$ for the Poisson probability mass function. We must find an appropriate joint mass function for X and Y, i.e., an element Q of the vector space $\mathbb{R}^{\mathbb{N}^2}$ with all components non-negative and satisfying

$$\sum_j Q_{i,j} = \pi_i^{\alpha\lambda}\,, \quad \sum_i Q_{i,j} = \pi_j^{(1-\alpha)\lambda}\,, \quad \sum_i Q_{i,k-i} = \pi_k^{\lambda}\,, \tag{6}$$

and

$$Q_{0,1} = Q_{1,1} = Q_{2,0} = 0\,. \tag{7}$$

Let $P \in \mathbb{R}^{\mathbb{N}^2}$ be the mass function for independent Poisson random variables, i.e., $P_{i,j} := \pi_i^{\alpha\lambda}\pi_j^{(1-\alpha)\lambda}$, and note that P satisfies (6) (with P in place of

Q). For $s, t \in \mathbb{N}$ define $E^{s,t} \in \mathbb{R}^{\mathbb{N}^2}$ by $E^{s,t}_{i,j} := 0$ for $(i,j) \notin [s, s+2] \times [t, t+2]$, and

$E^{s,t}_{i,j} :=$

$i \diagdown j$	t	$t+1$	$t+2$
s	0	-1	1
$s+1$	1	0	-1
$s+2$	-1	1	0

and note that $\sum_j E^{s,t}_{i,j} = \sum_i E^{s,t}_{i,j} = \sum_i E^{s,t}_{i,k-i} = 0$.

Now let

$$Q := P + P_{0,1}E^{0,0} - (-P_{0,1} + P_{2,0})E^{1,0} - (-P_{0,1} + P_{2,0} + P_{1,1})E^{0,1}\,.$$

From the definition of Q, it is easy to verify that (7) holds. (The idea is that adding a multiple of $E^{s,t}$ moves mass from location $(s, t+1)$ to $(s+1, t)$, without affecting the locations (i,j) with $i + j \leq s + t$. First we transfer mass $P_{0,1}$ from location $(0,1)$ to $(1,0)$; this results in mass $P_{2,0} - P_{0,1}$ at $(2,0)$, which we then transfer to $(1,1)$; finally we similarly transfer the current mass at $(1,1)$ to $(0,2)$.) The equalities (6) follow from the above observations on sums involving P and E, so it remains only to check non-negativity of Q for λ sufficiently large. This follows on noting that for some $c = c(k, \alpha) > 0$ we have $P_{i',j'} \geq c\lambda P_{i,j}$ whenever $i + j = k$ and $i' + j' = k + 1$; therefore it suffices to take λ large enough compared with $c(1,\alpha)^{-1}, \ldots, c(4,\alpha)^{-1}$. □

For later convenience we next rephrase Lemma 12 in terms of a mapping that constructs X from $X + Y$.

Corollary 13. *For any $\alpha \in (0,1)$, there exist a $k(\alpha)$ and a measurable function $F : \mathbb{N} \times [0,1] \to \mathbb{N}$ with the following properties:*

(a) For all $(n,u) \in \mathbb{N} \times [0,1]$, we have that $F(n,u) \leq n$.

(b) For all $u \in [0,1]$, we have that $F(1,u) = 1$ and $F(2,u) = 0$.

(c) If $\bar{X}$ is a Poisson random variable with mean $\bar{\lambda} > k(\alpha)$ and U is a $\mathrm{U}[0,1]$ random variable independent of $\bar{X}$, then $F(\bar{X}, U)$ and $\bar{X} - F(\bar{X}, U)$ are Poisson random variables with means $\alpha\bar{\lambda}$ and $(1-\alpha)\bar{\lambda}$ respectively.

Proof. Let $\alpha \in (0,1)$ and $k(\alpha)$ be as in Lemma 12. Let $\bar{X}$ be a Poisson random variable with mean $\bar{\lambda} > k(\alpha)$ and let U be a $\mathrm{U}[0,1]$ random variable independent of $\bar{X}$. By Lemma 12, let X and Y be Poisson random variables with means $\alpha\lambda$ and $(1-\alpha)\lambda$ such that $X + Y \stackrel{d}{=} \bar{X}$. Define F so that

$$(\bar{X}, F(\bar{X}, U)) \stackrel{d}{=} (X + Y, X)\,. \qquad \square$$

With Corollary 13 the proof of Proposition 8 is relatively straightforward, except that property (e) requires a little care. We next present some definitions and elementary facts about Poisson processes that will be useful in the proof and in the rest of the paper.

Recall that for $\mu \in \mathbb{M}$, we denote the restriction of μ to a set $A \in \mathcal{B}$ via:

$$\mu\big|_A(\cdot) := \mu(\cdot \cap A)\,.$$

Recall that $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. We say that the inter-point distances of a point measure $\mu \in \mathbb{M}$ are distinct if for all $x, y, u, v \in [\mu]$ such that $\{x, y\} \neq \{u, v\}$ and $x \neq y$, we have that $\|x - y\| \neq \|u - v\|$.

Lemma 14 (Elementary facts about Poisson point processes)**.** *Let X be a Poisson point process on $\mathbb{R}^d$ with positive intensity and law P.*

(a) Let $a \in \mathbb{R}^d$. The distances from the X-points to the point a are distinct P-a.s.

(b) For all $d \geq 1$, the inter-point distances of X are distinct P-a.s.

(c) P-a.s., every set of d elements of $[X]$ has linear span equal to all of $\mathbb{R}^d$.

Proof. The proof follows easily from Lemma 7. □

Proof of Proposition 8. Let X be a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$. Let $\alpha := \lambda/\lambda'$ and let $k(\alpha)$ be defined as in Corollary 13. Let $K > 0$ be so that $\bar{\lambda} := K\lambda > k(\alpha)$. Let $A \in \mathcal{B}$ have Lebesgue measure larger than K. Let $\bar{X} := X(A)$, so that $\bar{X}$ is a Poisson random variable. Let F be a function as in Corollary 13. Let U be a $\mathrm{U}[0,1]$ random variable independent of X. Let $g_1, g_2 : [0,1] \to [0,1]$ be two functions as in Lemma 5 so that $U_1 := g_1(U)$ and $U_2 := g_2(U)$ are independent $\mathrm{U}[0,1]$ random variables. Note that by property (a) of Corollary 13, $F(X(A), U_1) \leq X(A)$. We shall define ϕ_A^{fin} so that $[\phi_A^{\mathrm{fin}}(X, U)]$ is a subset of $[X|_A]$ of size $F(X(A), U_1)$. Moreover, conditional on $F(X(A), U_1) = j$, each subset of $[X|_A]$ of size j will be chosen uniformly at random using the randomness provided by U_2. To do this carefully, we shall tag the points in $[X|_A]$ and specify a way to use the randomness provided by U_2.

Consider the following enumeration of the points in $[\mu|_A]$. The ***center of mass*** of a Borel set C with positive finite Lebesgue measure $\mathcal{L}(C) > 0$ is given by

$$\frac{1}{\mathcal{L}(C)} \int_C x \, d\mathcal{L}(x) \in \mathbb{R}^d\,. \tag{8}$$

Let a be the center of mass of A. We say that μ admits the ***centric enumeration*** on A if $\mu(A) > 0$ and if the distances from a to the points in $[\mu|_A]$ are distinct. The centric enumeration on A is given by the bijection $\iota = \iota_\mu : [\mu|_A] \to \{1, 2, \ldots, \mu(A)\}$, where $\iota(x) < \iota(y)$ iff $\|x - a\| < \|y - a\|$. Note that by Lemma 14, part (a), X admits the centric enumeration on A P_λ-a.s. when $X(A) > 0$.

Now we define some auxiliary functions that, when composed with $\mathrm{U}[0, 1]$ random variables, yield random variables with certain distributions. For any set B, let $\boldsymbol{\mathcal{P}}(B)$ denote the set of all subsets of B. Let $\{s_{i,j}\}_{j \le i}$ be a collection of measurable functions where $s_{i,j} : [0, 1] \to \boldsymbol{\mathcal{P}}(\{1, 2, \ldots i\})$ and if U' is a $\mathrm{U}[0, 1]$ random variable, then $s_{i,j}(U')$ is uniformly distributed over subsets of size j of $\{1, 2 \ldots, i\}$.

For all $\mu \in \mathbb{M}$ that do not admit the centric enumeration on A, set $\phi_A^{\mathrm{fin}}(\mu, u) = 0$ for all $u \in [0, 1]$. Otherwise, for $(\mu, u) \in \mathbb{M} \times [0, 1]$, we proceed as follows. If $\mu(A) = i$, let $\iota : [\mu|_A] \to \{1, \ldots, i\}$ be the centric enumeration. Let $F(i, g_1(u)) = j$. Define $\phi_A^{\mathrm{fin}}(\mu, u)$ to be the simple point measure with support $\{x \in [\mu|_A] : \iota(x) \in s_{i,j}(u)\}$.

Clearly, by definition, ϕ_A^{fin} is monotone and $\phi_A^{\mathrm{fin}}(\mu, u) = \phi_A^{\mathrm{fin}}(\mu|_A, u)$. From Corollary 13, property (c), it is immediate that $\phi_A^{\mathrm{fin}}(X, U)(A)$ and $X(A) - \phi_A^{\mathrm{fin}}(X, U)(A)$ are Poisson random variables with means $\lambda' \mathcal{L}(A)$ and $(\lambda - \lambda')\mathcal{L}(A)$ respectively. Moreover it is easy to check with the help of Lemma 7 that in fact $\phi_A^{\mathrm{fin}}(X, U)$ and $X\big|_A - \phi_A^{\mathrm{fin}}(X, U)$ are Poisson point processes on A with intensities λ' and $\lambda - \lambda'$ respectively. Thus properties (c), (a) and (b) all hold. It is easy to see that property (d) is also inherited from property (b) of Lemma 13. Moreover we have the required property (e) since we enumerated the points in the support of $\mu|_A$ in an isometry-equivariant way via the centric enumeration, while the functions $g_1, g_2, f_i, s_{i,j}$ are fixed functions independent of μ and A. $\square$

5 Selection Rules

We shall now define an important class of isometry-equivariant partitions that will have a certain independence property. Recall that the open ***ball*** of radius r centered at x is denoted by $B(x, r) := \big\{y \in \mathbb{R}^d : \|x - y\| < r\big\}$. The closed ball is denoted by $\bar{B}(x, r) := \big\{y \in \mathbb{R}^d : \|x - y\| \le r\big\}$. Let $\mathfrak{F} \subset \mathcal{B}$ denote the set of closed subsets of $\mathbb{R}^d$. An ***R-selection rule*** is a mapping $\Psi : \mathbb{M} \to \mathfrak{F}$ that has the following properties.

(a) If X is a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$ and law P_λ, then P_λ-a.s. $\Psi(X)$ is a non-empty union of disjoint closed balls of radius R.

(b) The map Ψ is isometry-equivariant; that is, for all isometries θ of $\mathbb{R}^d$ and all $\mu \in \mathbb{M}$, we have that $\Psi(\theta\mu) = \theta\Psi(\mu)$.

(c) For all $\mu, \mu' \in \mathbb{M}$, provided μ and μ' agree on the set

$$H(\mu) = H_\Psi(\mu) := \left(\bigcup_{x \in \Psi(\mu)} \bar{B}(x, 2) \right)^c, \tag{9}$$

we have that $\Psi(\mu) = \Psi(\mu')$.

(d) The map Ψ is measurable; see below for the precise meaning of this.

Let Ψ be an R-selection rule and let $\mu \in \mathbb{M}$. We call the connected components of $\Psi(\mu)$ the ***globes*** (under μ) and we denote the set of globes by $\mathrm{Globes}[\Psi(\mu)]$. The ***ether*** is $\Psi(\mu)^c := \mathbb{R}^d \setminus \Psi(\mu)$. Note that the set of globes together with the ether form an isometry-equivariant partition of $\mathbb{R}^d$.

The idea behind the key condition (c) is that $\Psi(\mu)$ is determined only by the restriction of μ to $\Psi(\mu)^c$ (for technical reasons it is convenient to insist that it is determined even on the smaller set $H(\mu) \subset \Psi(\mu)^c$). This will have the consequence that for a Poisson point process X, conditional on $\Psi(X)$, the process restricted to $\Psi(X)$ is still a Poisson point process.

Proposition 15. *For all $d \geq 1$ and all $R > 0$, there exists an R-selection rule.*

We postpone the construction of selection rules until Section 6. Sometimes when the value of R is not important we shall refer to Ψ simply as a selection rule. The key property of selection rules is the following.

Proposition 16 (Key equality)**.** *Let X and W be independent Poisson point process on $\mathbb{R}^d$ with the same intensity. For a selection rule Ψ, the process $Z := W\big|_{\Psi(X)} + X\big|_{\Psi(X)^c}$ has the same law as X, and $\Psi(X) = \Psi(Z)$.*

Proposition 16 states that conditional on $\Psi(X)$, not only is $X|_{\Psi(X)}$ a Poisson point process on $\Psi(X)$, it is also independent of $X|_{\Psi(X)^c}$.

Some Remarks on Measurability. It will be obvious from our construction of selection rules that measurability will not be an issue. However, for the sake of completeness and since we want to prove Proposition 16 before providing the explicit construction of selection rules, we assign the ***Effros*** σ-algebra to $\mathfrak{F}$. For each compact set $K \in \mathcal{B}$, let $\mathfrak{F}_K := \{F \in \mathfrak{F} : F \cap K \neq \emptyset\}$. The Effros σ-algebra for $\mathfrak{F}$ is generated by the sets $\mathfrak{F}_K$ for all compact sets $K \in \mathcal{B}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We call a measurable function $\mathcal{X} : \Omega \to \mathfrak{F}$ a ***random closed set***. Thus if X is a Poisson point process and Ψ is a selection rule, then $\Psi(X)$ is a random closed set. We shall not need to use any results from the theory of random closed sets; we refer the interested reader to [16] for background.

Remarks on the Proof of Proposition 16. It is immediate from property (c) that $\Psi(X) = \Psi(Z)$. The isometry-equivariance of selection rules (property (b)) will not play a role in the proof of Proposition 16. For the purposes of the following discussion, let us assume that Ψ does not have to satisfy property (b). Temporarily suppose instead that Ψ satisfies the following additional requirement.

(b′) There exists a *fixed* Borel set D such that if X is a Poisson point process on $\mathbb{R}^d$, then $\Psi(X) \subset D \subset H_\Psi(X)^c$.

By property (c) in the definition of a selection rule, it is easy to see that $\Psi(X)$ is $\sigma(X|_{D^c})$-measurable. Since $X|_D$ and $X|_{D^c}$ are independent, we have that

$$W\big|_{D\cap\Psi(X)} + X\big|_{D\cap\Psi(X)^c} \stackrel{d}{=} X\big|_D ,$$

where $W \stackrel{d}{=} X$ and W is independent of X. Moreover, one can verify (see Lemma 18 below) that

$$W\big|_{D\cap\Psi(X)} + X\big|_{D\cap\Psi(X)^c} + X\big|_{D^c} \stackrel{d}{=} X. \tag{10}$$

If Ψ satisfies condition (b′), then the left-hand side of (10) equals $W\big|_{\Psi(X)} + X\big|_{\Psi(X)^c}$ and Lemma 16 follows.

The above argument suggests that to prove Proposition 16, we should examine events where $\Psi(X)$ is contained in some deterministic set. However in general, such events will have probability zero. We can overcome this problem by considering events where for some bounded Borel set A, we have

that $\Psi(X) \cap A$ is contained in some deterministic set. For each bounded Borel set A, Lemma 17 below specifies some additional useful properties that we require of such events.

Lemma 17. *Let X be a Poisson point process on $\mathbb{R}^d$ with positive intensity. Let Ψ be an R-selection rule and let H be defined as in (9). Let A be a bounded Borel set. There exists a finite set F, a collection of disjoint events $\{E(\alpha)\}_{\alpha \in F}$, and a collection of bounded Borel sets $\{D(\alpha)\}_{\alpha \in F}$ with the following properties:*

(i) For all $\alpha \in F$, on the event $E(\alpha)$, we have that

$$\Psi(X) \cap A \subset D(\alpha) \subset H(X)^c .$$

(ii) For all $\alpha \in F$, the event $E(\alpha)$ is $\sigma(X\big|_{D(\alpha)^c})$-measurable.

(iii) The disjoint union $\bigcup_{\alpha \in F} E(\alpha)$ is an event of probability one.

We shall prove this later.

The following lemma will be useful in the proof of Proposition 16. In particular, it justifies equation (10) when Ψ satisfies condition (b′). The lemma is a technical generalization of the fact if X and W are two independent Poisson point processes on $\mathbb{R}^d$ with the same intensity, then for all $s \in \mathcal{B}$ we have

$$W|_s + X|_{s^c} \stackrel{d}{=} X. \tag{11}$$

Lemma 18. *Let $\mathcal{X}$ and $\mathcal{W}$ be Poisson point processes with the same intensity on some Borel set $\mathcal{D} \subset \mathbb{R}^d$. Let $\mathcal{T}$ be a random closed set and let $\mathcal{S} := \mathcal{T} \cap \mathcal{D}$. Let $\mathcal{Y}$ be any point process and let $\mathcal{V}$ be an event. Let $\mathcal{S}' := \mathcal{D} \setminus \mathcal{S}$. If $\mathcal{W}$ is independent of $(\mathcal{X}, \mathcal{S}, \mathcal{Y}, \mathcal{V})$ and if $\mathcal{X}$ is independent of $(\mathcal{S}, \mathcal{Y}, \mathcal{V})$, then for all measurable sets of point measures $\mathcal{A} \in \mathcal{M}$,*

$$\mathbb{P}\Big(\{\mathcal{X} + \mathcal{Y} \in \mathcal{A}\} \cap \mathcal{V} \Big) = \mathbb{P}\Big(\{\mathcal{W}\big|_{\mathcal{S}} + \mathcal{X}\big|_{\mathcal{S}'} + \mathcal{Y} \in \mathcal{A}\} \cap \mathcal{V} \Big) .$$

Proof. Let $\mu_{\mathcal{X}}$ be the law of $\mathcal{X}$ and let Q be the joint law of $\mathcal{S}, \mathcal{Y}$ and $\mathbf{1}_{\mathcal{V}}$. From (11) it is easy to see that for all Borel $s \subset \mathcal{D}$ and for all $\mathcal{A}' \in \mathcal{M}$,

$$\begin{aligned} \int \mathbf{1}_{[x|_s + x|_{s'} \in \mathcal{A}']} \, d\mu_{\mathcal{X}}(x) &= \mathbb{P}(\mathcal{X} \in \mathcal{A}') \\ &= \mathbb{P}(\mathcal{W}|_s + \mathcal{X}|_{s'} \in \mathcal{A}') \\ &= \iint \mathbf{1}_{[w|_s + x|_{s'} \in \mathcal{A}']} \, d\mu_{\mathcal{X}}(w) \, d\mu_{\mathcal{X}}(x). \end{aligned} \tag{12}$$

Let $\mathcal{A} \in \mathcal{M}$ and $L := \mathbb{P}\Big(\{\mathcal{X} + \mathcal{Y} \in \mathcal{A}\} \cap \mathcal{V}\Big)$. By the independence of $\mathcal{X}$ and $(\mathcal{S}, \mathcal{Y}, \mathcal{V})$, we have that

$$\begin{aligned} L &= \iint \mathbf{1}_{[x+y\in\mathcal{A}]} v \, d\mu_{\mathcal{X}}(x) dQ(s,y,v) \\ &= \int \Big(\int \mathbf{1}_{[x|_s + x|_{s'} + y \in \mathcal{A}]} \, d\mu_{\mathcal{X}}(x) \Big) v \, dQ(s,y,v). \end{aligned} \tag{13}$$

Applying (12) to (13), we obtain that

$$L = \iiint \mathbf{1}_{[w|_s + x|_{s'} + y \in \mathcal{A}]} v \, d\mu_{\mathcal{X}}(w) d\mu_{\mathcal{X}}(x) \, dQ(s,y,v) \,. \tag{14}$$

Since $\mathcal{X}$ and $\mathcal{W}$ are independent and $(\mathcal{X}, \mathcal{W})$ and $(\mathcal{S}, \mathcal{Y}, \mathcal{V})$ are independent, we easily recognize that the right-hand side of equation (14) is equal to $\mathbb{P}\Big(\{\mathcal{W}\big|_{\mathcal{S}} + \mathcal{X}\big|_{\mathcal{S}'} + \mathcal{Y} \in \mathcal{A}\} \cap \mathcal{V}\Big)$. □

With the help of Lemmas 17 and 18 we now prove Proposition 16.

Proof of Proposition 16. Let $X, W : \Omega \to \mathbb{M}$ be independent Poisson point processes on $\mathbb{R}^d$ with the same intensity, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall use ω to denote an element of the probability space, and during this proof $X(\omega)$ will denote the point measure that is the image of ω under the random variable X (not "the number of X-points in ω"). Let Ψ be an R-selection rule and let $Z := W\big|_{\Psi(X)} + X\big|_{\Psi(X)^c}$. Let $\mathcal{A} \in \mathcal{M}$. It suffices to show that $\mathbb{P}\left(X|_A \in \mathcal{A}\right) = \mathbb{P}\left(Z|_A \in \mathcal{A}\right)$ for all bounded Borel sets A. Let A be a bounded Borel set and let $\{E(\alpha)\}_{\alpha\in F}$ and $\{D(\alpha)\}_{\alpha\in F}$ be collections of events and subsets of $\mathbb{R}^d$ that satisfy the conditions of Lemma 17. We shall show that for all $\alpha \in F$,

$$\mathbb{P}\big(\{X|_A \in \mathcal{A}\} \cap E(\alpha)\big) = \mathbb{P}\big(\{Z|_A \in \mathcal{A}\} \cap E(\alpha)\big) \,. \tag{15}$$

By summing over all $\alpha \in F$, we can then conclude by property (iii) of Lemma 17 that $\mathbb{P}\left(X|_A \in \mathcal{A}\right) = \mathbb{P}\left(Z|_A \in \mathcal{A}\right)$. Let us fix $\alpha \in F$ and set $E := E(\alpha)$ and $D := D(\alpha)$. Observe that for all $\omega_1, \omega_2 \in E$, we have $\Psi(X(\omega_1)) = \Psi(X(\omega_2))$ whenever $X(\omega_1) = X(\omega_2)$ on D^c. This follows from property (c) in the definition of a selection rule and property (i) of Lemma 17. Now define $S := \Psi(X|_{D^c})$. Clearly, S is $\sigma(X|_{D^c})$-measurable and on the event E, we have that $S = \Psi(X)$. Since X is a Poisson point process, we

have that $X|_{D\cap A}$ is independent of $X|_{D^c\cap A}$. Also, by property (ii) of Lemma 17 we have that $E \in \sigma(X|_{D^c})$. By applying Lemma 18 with the following substitutions:

$$\mathcal{D} = D \cap A\,, \quad \mathcal{X} = X|_{D\cap A}\,,\ \mathcal{W} = W|_{D\cap A}\,,$$
$$\mathcal{T} = S\,, \quad \mathcal{S} = S \cap D \cap A\,,\ \mathcal{Y} = X|_{D^c\cap A}\,, \quad \mathcal{V} = \mathbf{1}_E\,,$$

it is easy to check that

$$\mathbb{P}\big(\{X|_A \in \mathcal{A}\} \cap E\big) = \mathbb{P}\big(\{W|_{D\cap A\cap S} + X|_{D\cap A\cap S^c} + X|_{D^c\cap A} \in \mathcal{A}\} \cap E\big)\,.$$

Thus from the definition of S and property (i) of Lemma 17, we have that

$$\mathbb{P}\big(\{X|_A \in \mathcal{A}\} \cap E\big) = \mathbb{P}\big(\big\{W|_{\Psi(X)\cap A} + X|_{\Psi(X)^c\cap A} \in \mathcal{A}\big\} \cap E\big)\,.$$

By the definition of Z, we see that we have verified equation (15) as required. □

It remains to prove Lemma 17.

Proof of Lemma 17. We need some preliminary definitions. The open ***cube*** of side length $2r$ centered at the origin is the set $(-r, r)^d$. The ***diameter*** of a set $A \subset \mathbb{R}^d$ is $\sup_{x,y\in A} \|x-y\|$. Let X be a Poisson point process on $\mathbb{R}^d$ and let Ψ be an R-selection rule. Recall that by property (a) in the definition of a selection rule, all globes are balls of radius R. Fix a bounded Borel set A. Let $A' := \bigcup_{x\in A} B(x, 2R)$. Let $\{c_i\}_1^N$ be a collection of disjoint cubes of diameter $\frac{1}{2}$ such that their union contains the set A'. Thus, some cubes may not be open. Let $a_i \in c_i$ be the centers of the cubes. Let F_i be the event that the center of some globe (under X) is an element of the cube c_i. For a binary sequence $\alpha \in \{0,1\}^N$ of length N, define

$$E(\alpha) := \Big(\bigcap_{\substack{1\le i\le N:\\ \alpha(i)=1}} F_i\Big) \cap \Big(\bigcap_{\substack{1\le i\le N:\\ \alpha(i)=0}} F_i^c\Big)\,.$$

Set $F := \Big\{\alpha \in \{0,1\}^N : \mathbb{P}(E(\alpha)) > 0\Big\}$. Note that the events $\{E(\alpha)\}_{\alpha\in F}$ are disjoint and their union over all α is an event of probability 1, so that condition (iii) is satisfied. Note that if $x \in \mathbb{R}^d$ and $\|x - a_i\| \le \frac{1}{2}$, then

$$\bar{B}(x, R) \subset B(a_i, R+1) \subset \bar{B}(x, R+2)\,. \tag{16}$$

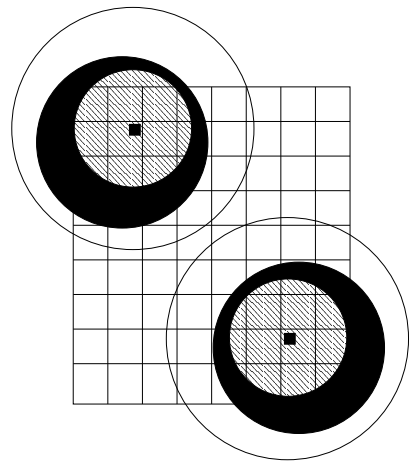

Figure 2: The grid is an illustration of the cubes c_i. The black squares are the centers of the globes. The hatched discs are the globes, the union of the shaded discs is the set $D(\alpha)$ that contains the globes intersecting A, and the area contained in the largest circles is part of the set $H(X)^c$.

Define

$$D(\alpha) := \bigcup_{\substack{1 \le i \le N: \\ \alpha(i)=1}} B(a_i, R+1)\,.$$

Since every globe that intersects A has its center lying at distance at most R from A, every globe that intersects A must have a center in some cube c_i. By definition, for every $\alpha \in F$, on the event $E(\alpha)$ we see from (16) and (9) that

$$\Psi(X) \cap A \subset D(\alpha) \subset H(X)^c\,,$$

since the diameter of each cube c_i is $\frac{1}{2}$. See Figure 2 for an illustration.

Thus condition (i) is satisfied. Observe that for each $\alpha \in F$, we have that $E(\alpha) \in \sigma(X|_{D(\alpha)^c})$ by property (c) in the definition of a selection rule, so that condition (ii) is also satisfied. □

Proposition 16 will be instrumental in proving Theorems 1 and 2. In Corollary 20 below, we make an important step in this direction by constructing a splitting that involves different mechanisms on the globes and on the ether. Before stating this result, we need some preliminary definitions. In particular, recall the elementary fact that if each point of a Poisson point

process X with intensity λ is deleted independently of all others with probability $\frac{\lambda'}{\lambda}$, where $\lambda' < \lambda$, then the remaining points and the deleted points form independent Poisson point processes with intensities λ' and $\lambda - \lambda'$. To facilitate later variations on this theme, we shall give a very explicit version of this fact.

Sometimes it will be convenient to specify a well ordering of the sets $[\mu]$, $[\mu|_{\Psi(\mu)^c}]$ and Globes$[\Psi(\mu)]$. This can be done in the following way. Consider the ordering $\prec$ on $\mathbb{R}^d$ in which $x \prec y$ iff $\|x\| < \|y\|$ or iff $\|x\| = \|y\|$ and x is less than y in the lexicographic ordering of $\mathbb{R}^d$. Thus we can well order $[\mu]$ and $[\mu|_{\Psi(\mu)^c}]$ via $\prec$ and well order Globes$[\Psi(\mu)]$ by well ordering the centers of the globes via $\prec$. We shall call $\prec$ the ***radial ordering***.

Define $F^{\text{coin}} = F^{\text{coin}}_{(\lambda,\lambda')} : \mathbb{R}^d \times [0,1] \to \mathbb{M}$ via

$$F^{\text{coin}}(x,u) := \mathbf{1}_{[u \le \frac{\lambda'}{\lambda}]}\delta_x \,. \tag{17}$$

Define $\phi^{\text{ind}} = \phi^{\text{ind}}_{(\lambda,\lambda')} : \mathbb{M} \times [0,1] \to \mathbb{M}$ by

$$\phi^{\text{ind}}_{(\lambda,\lambda')}(\mu,u) := \sum_{i=1}^{\infty} F^{\text{coin}}_{(\lambda,\lambda')}(x_i, g_i(u)), \qquad (\mu,u) \in \mathbb{M} \times [0,1]\,, \tag{18}$$

where $\{x_i\}_{i=1}^{\infty}$ is $[\mu]$ ordered by $\prec$ and the g_i are from Lemma 5. We shall call ϕ^{ind} the ***standard splitting***. The following fact is elementary.

Lemma 19 (Independent splitting)**.** *If X is a Poisson point process on $\mathbb{R}^d$ with intensity λ and U is a $\mathrm{U}[0,1]$ random variable independent of X, then for all $A \in \mathcal{B}$ and for all $\lambda' < \lambda$ we have that $\phi^{\text{ind}}_{(\lambda,\lambda')}(X|_A, U)$ and $X|_A - \phi^{\text{ind}}_{(\lambda,\lambda')}(X|_A, U)$ are independent Poisson point processes on A with intensities λ' and $\lambda - \lambda'$ respectively.*

Corollary 20. *Let X be a Poisson point process on $\mathbb{R}^d$ with intensity λ and let $\lambda' < \lambda$. Let ϕ^{fin} be the splitting from Proposition 8. Let Ψ be an R-selection rule, where the Lebesgue measure of $B(\mathbf{0}, R)$ is larger than that of the constant $K(\lambda,\lambda')$ from Proposition 8. Let $\{b_i\}_{i\in\mathbb{Z}^+} = \text{Globes}[\Psi(X)]$, where we have ordered the globes via the radial ordering. Let U be a $\mathrm{U}[0,1]$ random variable independent of X and let $g_i : [0,1] \to [0,1]$ be a sequence of functions as in Lemma 5. The mapping $\Phi = \Phi_{(\lambda,\lambda')}$ defined by*

$$\Phi(X,U) := \sum_{i\in\mathbb{Z}^+} \phi^{\text{fin}}_{(b_i,\lambda,\lambda')}(X\big|_{b_i}, g_i(U)) + \phi^{\text{ind}}_{(\lambda,\lambda')}(X\big|_{\Psi(X)^c}, g_0(U))$$

is a splitting such that $\Phi(X,U)$ and $X-\Phi(X,U)$ are Poisson point processes with intensities λ' and $\lambda-\lambda'$.

Proof. The inequality $\Phi(X,U)\le X$ is obvious from the definition of Φ, so we just need to check that $\Phi(X,U)$ and $X-\Phi(X,U)$ have the right distributions. This is made possible via Proposition 16. Let W be a Poisson point process on $\mathbb{R}^d$ with intensity λ that is independent of X and U. Let U_1, U_2 be independent $\mathrm{U}[0,1]$ random variables that are also independent of X and W. From the definition of ϕ^{ind}, it is easy to see that

$$\phi^{\mathrm{ind}}_{(\lambda,\lambda')}(W|_{\Psi(X)}+X|_{\Psi(X)^c},U_1)\stackrel{d}{=}\phi^{\mathrm{ind}}_{(\lambda,\lambda')}(W\big|_{\Psi(X)},U_1)+\phi^{\mathrm{ind}}_{(\lambda,\lambda')}(X\big|_{\Psi(X)^c},U_2)\,,$$

since the ordering of the points of $W|_{\Phi(X)}+X|_{\Phi(X)^c}$ is irrelevant as long as the ordering is independent of U_1 and U_2. By Proposition 16, we have that $X\stackrel{d}{=}W|_{\Phi(X)}+X|_{\Phi(X)^c}$, so we obtain that

$$\phi^{\mathrm{ind}}_{(\lambda,\lambda')}(X,U_1)\stackrel{d}{=}\phi^{\mathrm{ind}}_{(\lambda,\lambda')}(W\big|_{\Psi(X)},U_1)+\phi^{\mathrm{ind}}_{(\lambda,\lambda')}(X\big|_{\Psi(X)^c},U_2)\,. \tag{19}$$

From property (c) of Proposition 8 and Lemma 19, it is easy to see that for any $A\in\mathcal{B}$ with Lebesgue measure larger than K, we have

$$\phi^{\mathrm{fin}}_A(W|_A,U_1)\stackrel{d}{=}\phi^{\mathrm{ind}}(W|_A,U_1)\,.$$

Moreover, since X and W are independent, it follows that

$$\mathbb{P}\Big(\sum_{i\in\mathbb{Z}^+}\phi^{\mathrm{fin}}_{b_i}(W|_{b_i},g_i(U_1))\in\cdot\;\Big|\;X\Big)=\mathbb{P}\big(\phi^{\mathrm{ind}}(W|_{\Psi(X)},U_1)\in\cdot\;\big|\;X\big)\,. \tag{20}$$

(Recall that $\{g_i(U)\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. $\mathrm{U}[0,1]$ random variables.) Clearly by Proposition 16 and the definition of Φ, we have

$$\begin{aligned}\Phi(X,U)&\stackrel{d}{=}\Phi(W|_{\Psi(X)}+X|_{\Psi(X)^c},U)\\&\stackrel{d}{=}\sum_{i\in\mathbb{Z}^+}\phi^{\mathrm{fin}}_{b_i}(W|_{b_i},g_i(U_1))\;+\;\phi^{\mathrm{ind}}(X|_{\Psi(X)^c},U_2)\,.\end{aligned}$$

From equation (20) and the fact that X and W are independent, it is easy to verify that

$$\Phi(X,U)\stackrel{d}{=}\phi^{\mathrm{ind}}(W\big|_{\Psi(X)},U_1)+\phi^{\mathrm{ind}}(X|_{\Psi(X)^c},U_2)\,. \tag{21}$$

Putting (19) and (21) together, we obtain that $\Phi(X,U) \stackrel{d}{=} \phi^{\text{ind}}_{(\lambda,\lambda')}(X,U_1)$. Thus from Lemma 19 we have verified that $\Phi(X,U)$ is a Poisson point process of intensity λ'.

The proof that $X - \Phi(X,U)$ is a Poisson point process of intensity $\lambda - \lambda'$ follows by the same argument since ϕ^{fin} is a splitting by Proposition 8 and ϕ^{ind} is a splitting by Lemma 19. □

Let us remark that for Corollary 20, in order for Φ to be a splitting we must apply the splitting ϕ^{fin} in all the globes and not just the special ones. For example, if X is a Poisson point process on a bounded Borel set B, the following procedure will not result in a splitting: apply ϕ^{fin} if there are exactly one or two X points, otherwise apply ϕ^{ind}.

Before we begin the proof of Theorem 1, we first provide a construction of selection rules along with some other minor constructions that will be needed.

6 Construction of Selection Rules

Fix $d \geq 1$ and $R > 0$. We shall now construct an R-selection rule. We need some preliminary definitions. Recall the definition of the ***shell***

$$A(x;s,r) := \left\{ y \in \mathbb{R}^d : s \leq \|x-y\| \leq r \right\}.$$

Let X be a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$ and law P_λ. A point $x \in \mathbb{R}^d$ is called a ***pre-seed*** if $B(x, R+100+d)$ has the following two properties:

(a) $X\big(A(x; R+90+d, R+100+d)\big) = 0$;

(b) for every open ball B of radius $\frac{1}{2}$ satisfying $B \subset A(x; R+80, R+90+d)$, we have $X(B) \geq 1$.

Given $\mu \in \mathbb{M}$, we also say that x is pre-seed under μ if (a) and (b) hold with X replaced by μ. If x is a pre-seed, we call $A(x; R+90+d, R+100+d)$ the associated ***empty shell*** and $A(x; R+80, R+90+d)$ the associated ***halo***. Clearly pre-seeds exist P_λ-a.s. An R-selection rule will be defined so that its globes will be balls of radius R contained in $B(x, R+80)$ for some pre-seed x. See Figure 3 for an illustration of a pre-seed.

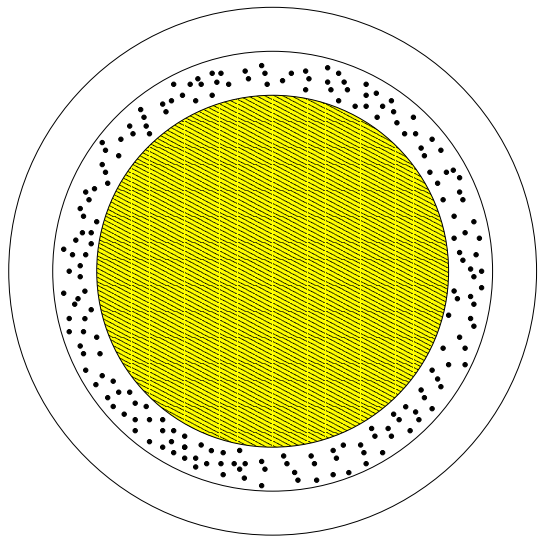

Figure 3: An illustration of a pre-seed. The outer ring (the empty shell) contains no X-points. The intermediate ring (the halo) is relatively densely filled with X-points. The shaded area is unspecified in terms of X.

Observe that if $x, y \in \mathbb{R}^d$ are pre-seeds, then $\|x-y\| \notin (2, 2(R+88+d))$; otherwise the empty shell of one pre-seed would intersect the halo of the other in such a way as to contradict the definition of a pre-seed. We say that two pre-seeds x, y are ***related*** if $\|x-y\| \leq 2$. This gives an equivalence relation on the pre-seeds.

Lemma 21. *If X is a Poisson point process on $\mathbb{R}^d$ with positive intensity and law P, then every equivalence class of pre-seeds under X is an open set with positive finite Lebesgue measure P-a.s.*

Proof. Let C be an equivalence class of pre-seeds under X. Let $a \in C$. We shall find an open ball centered at a that is contained in C. Define the distance between two sets A, B in the standard way: $\|A - B\| := \inf_{x \in A, y \in B} \|x - y\|$. Define the following:

$$\begin{aligned}
\delta_1 &:= \|A(a; R+90+d, R+100+d) - [X]\|\,, \\
\delta_2 &:= \sup\big\{r : \exists x\ (B(x,r) \subset A(a; R+80, R+90+d) \\
&\qquad\qquad \text{and } X\big(B(x,r)\big) = 0)\big\}\,, \\
\delta &:= \min(\delta_1, \delta_2, \frac{1}{2} - \delta_2)/4\,.
\end{aligned}$$

In words, δ_1 is the distance from the empty shell to the nearest X-point. Clearly, $\delta_1 \in (0,1)$. Also, δ_2 is the radius of the largest ball inside the halo that contains no X-point. Clearly $\delta_2 \in (0, \frac{1}{2})$. We easily see that for all $x \in \mathbb{R}^d$, if $\|x - a\| < d$, then $x \in C$. $\square$

We next choose a representative from each equivalence class in an isometry-equivariant way. Let C be an equivalence class of pre-seeds under μ. If C has positive Lebesgue measure (which happens almost surely for the Poisson point process), then we take $c \in \mathbb{R}^d$ to be its center of mass and declare that c is a ***seed***. Note that since C is a set of diameter at most 2, it is easy to see that C is contained in some ball of radius 2 that also contains the center of mass, whenever the center of mass exists. Note that c might not be a pre-seed (but it has properties similar to a pre-seed).

If c is a seed (under μ), we call $\bar{B}(c, R)$ a ***globe*** (under μ). Define the mapping $\Psi_R : \mathbb{M} \to \mathfrak{F}$ by stipulating that for each $\mu \in \mathbb{M}$, the Borel set $\Psi_R(\mu)$ is the union of the set of globes under μ. Given any two seeds, it is easy to see that their globes do not intersect. Thus the definition of a globe given here is consistent with the definition of a globe given in Section 5.

Next, we show that for $R > 0$, the mapping Ψ_R is a selection rule, thus proving Proposition 15.

Lemma 22. *Let $\Psi = \Psi_R$ be the mapping defined above. For all $\mu, \mu' \in \mathbb{M}$, if $\mu = \mu'$ on $H(\mu) := \left(\bigcup_{x \in \Psi(\mu)} \bar{B}(x, 2)\right)^c$, then μ and μ' have the same pre-seeds.*

Proof. Assume that μ and μ' agree on $H(\mu)$. Let $z \in \mathbb{R}^d$ be a pre-seed under μ. We claim that

$$\mu\big|_{A(z;R+80,R+100+d)} = \mu'\big|_{A(z;R+80,R+100+d)},$$

from which we deduce that z is also a pre-seed under μ'.

Let $C(z)$ be the equivalence class of pre-seeds to which z belongs. Let c be the center of mass of $C(z)$, so that c is a seed under μ. Since c has distance at most 4 from z and z has distance at least $2(R+88+d)$ from any pre-seed (under μ) not in $C(z)$, we have that c has distance at least $2(R+88+d)-4$ from any pre-seed (under μ) not in $C(z)$. Let $m > 0$ be the minimal distance from c to another seed (under μ). Clearly $m \geq 2(R+88+d)-8$. Since $\mu = \mu'$ on $H(\mu)$, we have that $\mu\big|_{A(c;R+2,m-R-2)} = \mu'\big|_{A(c;R+2,m-R-2)}$. Since z has distance at most 4 from c, clearly $\mu\big|_{A(z;R+80,R+100+d)} = \mu'\big|_{A(z;R+80,R+100+d)}$.

Therefore, $H(\mu') \subseteq H(\mu)$. Since $\mu = \mu'$ on $H(\mu')$, our claim, and the result, follow. $\square$

Proof of Proposition 15. Let $R > 0$ and $d \geq 1$. We shall now check that $\Psi = \Psi_R$ is indeed an R-selection rule.

Property (a): Let P be the law of a Poisson point process on $\mathbb{R}^d$ with positive intensity. Note that pre-seeds occur P-a.s. By Lemma 21 we have that for P-a.e. $\mu \in \mathbb{M}$, every equivalence class of pre-seeds under μ has a well-defined center of mass, which in turn is a seed. Therefore, we have that seeds occur P-a.s. and $\Psi(\mu) \neq \emptyset$ for P-a.e. μ. Also, by definition, if $\Psi(\mu) \neq \emptyset$, then $\Psi(\mu)$ is a disjoint union of balls of radius R.

Property (b): Let θ be an isometry of $\mathbb{R}^d$. If $x \in \mathbb{R}^d$ is a pre-seed under μ, then $\theta(x)$ is a pre-seed under $\theta(\mu)$. Therefore if C is an equivalence class of pre-seeds under $\mu \in \mathbb{M}$, then $\theta(C)$ is an equivalence class of pre-seeds under $\theta(\mu)$. If c is the center of mass of C, then $\theta(c)$ is the center of mass of $\theta(C)$. Hence if b is globe under μ, then $\theta(b)$ is a globe under $\theta(\mu)$. So clearly, Ψ is isometry-equivariant.

Property (c): Let $\mu, \mu' \in \mathbb{M}$ and assume that $\mu = \mu'$ on $H(\mu)$. By Lemma 22, μ and μ' have the same pre-seeds. Thus, they have the same seeds and hence the same globes. Therefore by the definition of Ψ, we have $\Psi(\mu) = \Psi(\mu')$. $\square$

7 Encoding and Distributing Randomness

Unfortunately, our proofs of Theorems 1 and 2 do not follow from Proposition 16 alone. Recall that in Examples 1 and 2 we partitioned $\mathbb{R}^d$ into cubes, and the cubes that contained exactly one or two Poisson points were special. The locations of the Poisson points in a special cube were converted into sequences of i.i.d. $\mathrm{U}[0,1]$ random variables whose elements were then assigned to the other cubes of the partition. The purpose of this section is to state a lemma that asserts the existence of a function that encapsulates the task of encoding and distributing randomness in the more complicated case where a deterministic partition is replaced by the selection rule from Section 6, and Example 2 is replaced by Theorem 1.

Let Ψ be a selection rule. We say that a globe under μ is ***one-special*** if it happens to contain exactly one μ-point, and ***two-special*** if it happens to contain exactly two μ-points. A globe is ***special*** if it is either one-special or two-special. Denote the set of one-special globes by $\mathrm{Globes}^1[\Psi(\mu)]$,

the set of two-special globes by $\text{Globes}^2[\Psi(\mu)]$ and the set of special globes by $\text{Globes}^{1,2}[\Psi(\mu)]$. Also let $\Psi^1(\mu), \Psi^2(\mu)$, and $\Psi^{1,2}(\mu)$ denote the union of the set of one-special, two-special and special globes respectively. Let $(\Psi^1(\mu))^c, (\Psi^2(\mu))^c$, and $(\Psi^{1,2}(\mu))^c$ denote the respective complements in $\mathbb{R}^d$. Note that by Proposition 16, if X is a Poisson point process on $\mathbb{R}^d$ with positive intensity and law P, then one-special globes and two-special globes exist under X P-a.s.

Lemma 23 (Assignment function)**.** *Let $d \geq 1$ and $R > 0$. Let $\Psi = \Psi_R$ be the selection rule from Section 6. There exists a function $\boldsymbol{U} = \boldsymbol{U}_\Psi : \mathbb{M} \times (\mathfrak{F} \cup \mathbb{R}^d) \to [0,1]$ with the following properties.*

(a) Let X be a Poisson point process on $\mathbb{R}^d$ with positive intensity. Let $\{\kappa(X)_i\}_{i\in\mathbb{N}} := \text{Globes}[\Psi(X)] \cup [X|_{\Psi(X)^c}]$, where we have ordered the set using the radial ordering. (Recall that globes are ordered by their centers.) If $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. $\mathrm{U}[0,1]$ random variables that is independent of X, then

$$\begin{aligned} &\Big(X|_{(\Psi^1(X))^c}, \Psi^1(X), \Psi(X),\ \{\boldsymbol{U}(X, \kappa(X)_i)\}_{i\in\mathbb{N}}\Big) \\ \stackrel{d}{=}\ &\Big(X|_{(\Psi^1(X))^c}, \Psi^1(X), \Psi(X),\ \{U_i\}_{i\in\mathbb{N}}\Big) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\Big(X|_{(\Psi^2(X))^c}, \Psi^2(X), \Psi(X),\ \{\boldsymbol{U}(X, \kappa(X)_i)\}_{i\in\mathbb{N}}\Big) \\ \stackrel{d}{=}\ &\Big(X|_{(\Psi^2(X))^c}, \Psi^2(X), \Psi(X),\ \{U_i\}_{i\in\mathbb{N}}\Big). \end{aligned} \tag{23}$$

(b) The map $\boldsymbol{U}$ is isometry-invariant; that is, for all isometries θ of $\mathbb{R}^d$ and for all $(\mu, b) \in \mathbb{M} \times (\mathfrak{F} \cup \mathbb{R}^d)$, we have $\boldsymbol{U}(\mu, b) = \boldsymbol{U}(\theta(\mu), \theta(b))$.

We call $\boldsymbol{U}_\Psi$ the ***assignment function*** for the selection rule Ψ. Thus if X is a Poisson point process and $b \in \text{Globes}[\Psi(X)]$ or if $b \in [X|_{\Psi(X)^c}]$, then the assignment function assigns a $\mathrm{U}[0,1]$ random variable $\boldsymbol{U}(X, b)$ to b. Property (a) states that the $\mathrm{U}[0,1]$ random variables have a certain independence property; the values of X on both the one-special and two-special globes are needed to determine the values of the assignment function. The map that we shall define in the next section to prove Theorem 1 will use $\boldsymbol{U}$ to assign $\mathrm{U}[0,1]$ random variables to the globes and the points of the ether.

We shall see that property (a) makes proving Theorem 1 easy. Property (b) is necessary to ensure that the map that we define is isometry-equivariant.

Let us also remark that since by property (c) in the definition of a selection rule, $\Psi(X)$ depends only on $X|_{\Psi(X)^c} \subset X|_{(\Psi^{1,2}(X))^c}$, therefore the addition of $\Psi(X)$ in (22) and (23) is actually redundant. We now have all the tools we need to prove Theorem 1. We defer the proof of Lemma 23 to Section 9. Much of the proof is bookkeeping, but for property (a) we shall need to appeal to Proposition 16.

8 Proof of Theorem 1

We are now in a position to prove Theorem 1. First we give the definition of the mapping that satisfies the conditions of Theorem 1. Let X be a Poisson point process on $\mathbb{R}^d$ with intensity λ and let $\lambda' < \lambda$. Recall the definition of the splitting ϕ^{fin} from Section 2 (Proposition 8) and the definitions of F^{coin} and ϕ^{ind} from (17) and (18) of Section 6. Let $R = R(\lambda, \lambda') > 0$ be so that the Lebesgue measure of $B(\mathbf{0}, R)$ is larger than the constant $K(\lambda, \lambda')$ of Proposition 4. Let $\Psi = \Psi_R$ be the R-selection rule from Section 6 and let $\boldsymbol{U}$ be the assignment function from Lemma 23. Define $\Gamma = \Gamma_{(\lambda,\lambda')}$ as follows. For all $\mu \in \mathbb{M}$,

$$\Gamma(\mu) := \sum_{b \in \text{Globes}[\Psi(\mu)]} \phi^{\text{fin}}_{(b,\lambda,\lambda')}(\mu|_b, \boldsymbol{U}(\mu, b)) + \sum_{x \in \left[\mu|_{\Psi(\mu)^c}\right]} F^{\text{coin}}_{(\lambda,\lambda')}(x, \boldsymbol{U}(\mu, x)) \,. \quad (24)$$

Proof of Theorem 1. From the definition of Γ it is easy to check that it is isometry-equivariant; we need only recall that by Lemma 23, the assignment function $\boldsymbol{U}$ is isometry invariant and that the splitting ϕ^{fin} and selection rule Ψ are isometry-equivariant. Also it obvious that Γ is monotone, so it suffices to check that $\Gamma(X)$ and $X - \Gamma(X)$ are Poisson point processes on $\mathbb{R}^d$ with intensities λ' and $\lambda - \lambda'$ respectively.

Let U be a $\mathrm{U}[0,1]$ random variable independent of X and let $g_i : [0,1] \to [0,1]$ be the functions from Lemma 5. Let $\{b_i\}_{i \in \mathbb{Z}^+} = \text{Globes}[\Psi(X)]$, where we have ordered the globes via the radial ordering. Similarly, let $\{x_i\}_{i \in \mathbb{Z}^+} = [X|_{\Psi(X)^c}]$. Let Φ be the splitting defined in Corollary 20. Note that Φ is a version of Γ that uses randomness from U instead of from certain points of

X. By property (d) of Proposition 8 we have that

$$\begin{aligned}\Phi(X,U) &= \sum_{i\in\mathbb{Z}^+} \mathbf{1}_{[X(b_i)\neq 1]}\phi^{\text{fin}}_{b_i}\big(X|_{b_i}, \boldsymbol{U}(X,g_i(U))\big) + \\ &\qquad \phi^{\text{ind}}(X|_{\Psi(X)^c}, g_0(U)) \end{aligned} \tag{25}$$

and

$$\begin{aligned} X-\Phi(X,U) &= \sum_{i\in\mathbb{Z}^+} \mathbf{1}_{[X(b_i)\neq 2]}\Big(X|_{b_i} - \phi^{\text{fin}}_{b_i}\big(X|_{b_i}, \boldsymbol{U}(X,g_i(U))\big)\Big) + \\ &\qquad X|_{\Psi(X)^c} - \phi^{\text{ind}}(X|_{\Psi(X)^c}, g_0(U))\,. \end{aligned} \tag{26}$$

We shall show that $\Gamma(X) \stackrel{d}{=} \Phi(X,U)$ and $X-\Gamma(X) \stackrel{d}{=} X-\Phi(X,U)$. Set

$$\begin{aligned}\alpha &:= \sum_{i\in\mathbb{Z}^+} \phi^{\text{fin}}_{b_i}(X|_{b_i}, \boldsymbol{U}(X,b_i))\,, \\ \beta &:= \sum_{i\in\mathbb{Z}^+} F^{\text{coin}}(x_i, \boldsymbol{U}(X,x_i))\,, \\ \alpha' &:= \sum_{i\in\mathbb{Z}^+} \big(X|_{b_i} - \phi^{\text{fin}}_{b_i}(X|_{b_i}, \boldsymbol{U}(X,b_i))\big)\,, \end{aligned}$$

and

$$\beta' := X|_{\Psi(X)^c} - \sum_{i\in\mathbb{Z}^+} F^{\text{coin}}(x_i, \boldsymbol{U}(X,x_i))\,.$$

By definition,

$$\Gamma(X) = \alpha+\beta \quad \text{and} \quad X-\Gamma(X) = \alpha'+\beta'\,.$$

By property (d) of Proposition 8,

$$\alpha = \sum_{i\in\mathbb{Z}^+} \mathbf{1}_{[X(b_i)\neq 1]}\phi^{\text{fin}}_{b_i}(X|_{b_i}, \boldsymbol{U}(X,b_i))$$

and

$$\alpha' = \sum_{i\in\mathbb{Z}^+} \mathbf{1}_{[X(b_i)\neq 2]}\big(X|_{b_i} - \phi^{\text{fin}}_{b_i}(X|_{b_i}, \boldsymbol{U}(X,b_i))\big)\,.$$

By property (a) of Lemma 23, we have that

$$\alpha+\beta \stackrel{d}{=} \sum_{i\in\mathbb{Z}^+} \mathbf{1}_{[X(b_i)\neq 1]}\phi^{\text{fin}}_{b_i}(X|_{b_i}, g_{2i}(U)) + \sum_{i\in\mathbb{Z}^+} F^{\text{coin}}(x_i, g_{2i+1}(U))$$

and

$$\alpha' + \beta' \stackrel{d}{=} \sum_{i \in \mathbb{Z}^+} \mathbf{1}_{[X(b_i) \neq 2]} \big(X|_{b_i} - \phi^{\text{fin}}_{b_i}(X|_{b_i}, g_{2i}(U)) \big) + X|_{\Psi(X)^c} - \sum_{i \in \mathbb{Z}^+} F^{\text{coin}}(x_i, g_{2i+1}(U)) \,.$$

Thus, from the definition of Φ in Corollary 20, it is easy to see that

$$\alpha + \beta \stackrel{d}{=} \Phi(X, U)$$

and

$$\alpha' + \beta' \stackrel{d}{=} X - \Phi(X, U) \,.$$

Hence by Corollary 20, we have that $\Gamma(X)$ and $X - \Gamma(X)$ are Poisson point processes with intensities λ' and $\lambda - \lambda'$ respectively. □

9 The Assignment Function

In this section we shall prove Lemma 23. Many of the same tools will be useful again in the proof of Theorem 2. Recall that the assignment function contained within it the two tasks of generating and distributing uniform random variables. First we discuss how we generate uniform random variables.

The following lemma describes explicitly how we convert the position of a single X-point in a ball (which is a uniform random variable on the ball) into a single uniform random variable on $[0, 1]$. We need to be explicit to preserve equivariance.

Lemma 24 (Uniform random variables)**.** *For every $d \geq 1$ and $R > 0$, define $f_{\bar{B}(c,R)} : \bar{B}(c, R) \to [0, 1]$ via*

$$f_{\bar{B}(c,R)}(x) := \left(\frac{\|x - c\|}{R} \right)^d .$$

The collection of mappings $\left\{ f_{\bar{B}(c,R)} \right\}_{c \in \mathbb{R}^d}$ has the following properties.

1. *If V is a $\mathrm{U}[\bar{B}(c, R)]$ random variable, then $f_{\bar{B}(c,R)}(V)$ is a $\mathrm{U}[0, 1]$ random variable.*

2. *We have isometry-invariance; that is, for any isometry θ of $\mathbb{R}^d$, $f_{\bar{B}(c,R)}(x) = f_{\theta(\bar{B}(c,R))}(\theta(x))$ for all $x \in \bar{B}(c, R)$.*

Proof. Here we shall make good use of the fact that we are working with balls. Recall that the Lebesgue measure of a d-ball of radius R is given by $C(d)R^d$ for some fixed constant $C(d) > 0$ depending only on d. Let V be a uniform random variable on the ball $\bar{B}(c, R)$. Then for $0 \leq x \leq 1$,

$$\mathbb{P}\left(f_{\bar{B}(c,R)}(V) \leq x\right) = \mathbb{P}\left(\|V - c\| \leq Rx^{\frac{1}{d}}\right) = \frac{\mathcal{L}(\bar{B}(Rx^{\frac{1}{d}}))}{\mathcal{L}(\bar{B}(R))} = x. \qquad \square$$

Each globe or X-point not in a globe will be associated to a one-special globe and to a two-special globe. It will be necessary to allow more than one globe or X-point to be associated to each special globe. First we need to develop some infrastructure. Recall that by Lemma 5 a single uniform random variable can be used to generate a sequence of i.i.d. $\mathrm{U}[0, 1]$ random variables.

Encoding functions. We associate to every special globe a $[0, 1]$-valued sequence in the following way. Let $\left\{f_{\bar{B}(c,R)}\right\}_{c\in\mathbb{R}^d}$ and $\{g_i\}_{i\in\mathbb{N}}$ be the collections of functions from Lemmas 24 and 5 respectively. Let Ψ be an R-selection rule. For each $b \in \mathrm{Globes}^1[\Psi(\mu)]$, let x_b denote the unique μ-point in b and for each $b \in \mathrm{Globes}^2[\Psi(\mu)]$, let x_b^1 and x_b^2 be the two μ-points in b, where we take x_b^1 to be the one closest to the origin in a lexicographic ordering. Recall that $\oplus$ denotes addition modulo one. Let $h' = h'_\Psi : \mathbb{M} \times \mathfrak{F} \to [0, 1]$ and $h = h_\Psi : \mathbb{M} \times \mathfrak{F} \to [0, 1]^{\mathbb{N}}$ be defined as follows:

$$h'_\Psi(\mu, b) \quad := \quad \begin{cases} f_b(x_b) & \text{if } \mu \in \mathbb{M}, \quad b \in \mathrm{Globes}^1[\Psi(\mu)], \\ f_b(x_b^1) \oplus f_b(x_b^2) & \text{if } \mu \in \mathbb{M}, \quad b \in \mathrm{Globes}^2[\Psi(\mu)], \\ 0 & \text{if } \mu \in \mathbb{M}, \quad b \notin \mathrm{Globes}^{1,2}[\Psi(\mu)] \end{cases} \tag{27}$$

and

$$h_\Psi(\mu, b) := \{g_i(h'(\mu, b))\}_{i\in\mathbb{N}} \ . \tag{28}$$

We call h_Ψ the ***encoding function*** for the selection rule Ψ and we call h'_Ψ the ***simplified encoding function*** for the selection rule Ψ.

Lemma 25. *Let $d \geq 1$ and $R > 0$. Let Ψ be an R-selection rule. Both the encoding and the simplified encoding functions h, h' satisfy the following properties.*

(a) The maps h, h' are isometry-invariant; that is, for all isometries θ of $\mathbb{R}^d$ and for all $(\mu, b) \in \mathbb{M} \times \mathfrak{F}$, $h(\mu, b) = h(\theta(\mu), \theta(b))$ and $h'(\mu, b) = h'(\theta(\mu), \theta(b))$.

(b) *Let X be a Poisson point process on $\mathbb{R}^d$ with positive intensity. Let $\{b_i^1\}_{i\in\mathbb{N}} := \mathrm{Globes}^1[\Psi(X)]$ and $\{b_i^2\}_{i\in\mathbb{N}} := \mathrm{Globes}^2[\Psi(X)]$, where we have ordered the sets of one-special and two-special globes by the radial ordering. If $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. $\mathrm{U}[0,1]$ random variables that is independent of X, then*

$$\left(X|_{(\Psi^1(X))^c}, \Psi^1(X), \left\{h'(X, b_i^1)\right\}_{i\in\mathbb{N}}\right) \stackrel{d}{=} \left(X|_{(\Psi^1(X))^c}, \Psi^1(X), \{U_i\}_{i\in\mathbb{N}}\right)$$

and

$$\left(X|_{(\Psi^2(X))^c}, \Psi^2(X), \left\{h'(X, b_i^2)\right\}_{i\in\mathbb{N}}\right) \stackrel{d}{=} \left(X|_{(\Psi^2(X))^c}, \Psi^2(X), \{U_i\}_{i\in\mathbb{N}}\right).$$

Similarly, if $\{U_i'\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables, independent of X, where $U_1' \stackrel{d}{=} \{U_i\}_{i\in\mathbb{N}}$, then

$$\left(X|_{(\Psi^1(X))^c}, \Psi^1(X), \left\{h(X, b_i^1)\right\}_{i\in\mathbb{N}}\right) \stackrel{d}{=} \left(X|_{(\Psi^1(X))^c}, \Psi^1(X), \{U_i'\}_{i\in\mathbb{N}}\right)$$

and

$$\left(X|_{(\Psi^2(X))^c}, \Psi^2(X), \{h(X, b_i)\}_{i\in\mathbb{N}}\right) \stackrel{d}{=} \left(X|_{(\Psi^2(X))^c}, \Psi^2(X), \{U_i'\}_{i\in\mathbb{N}}\right).$$

To prove Lemma 25, we shall use the following corollary of Proposition 16:

Corollary 26. *Let X be a Poisson point process on $\mathbb{R}^d$ with positive intensity and let Ψ be an R-selection rule. Let $\{b_i^1\}_{i\in\mathbb{N}} := \mathrm{Globes}^1[\Psi(X)]$ and let $\{b_i^2\}_{i\in\mathbb{N}} := \mathrm{Globes}^2[\Psi(X)]$, where we have ordered the one-special and two-special globes using the radial ordering. Also let $c_i^1 \in b_i^1$ and $c_i^2 \in b_i^2$ be the centers of the one-special and two-special globes. If $\{V_i^1\}_{i\in\mathbb{N}}, \{V_i^{21}\}_{i\in\mathbb{N}}$, and $\{V_i^{22}\}_{i\in\mathbb{N}}$ are independent i.i.d. sequences of $\mathrm{U}[\bar{B}(\mathbf{0}, R)]$ random variables that are independent of X, then*

$$\left(X|_{(\Psi^1(X))^c}, \Psi^1(X), \left\{X|_{b_i^1}\right\}_{i\in\mathbb{N}}\right) \stackrel{d}{=} \left(X|_{(\Psi^1(X))^c}, \Psi^1(X), \left\{\delta_{V_i^1+c_i^1}\right\}_{i\in\mathbb{N}}\right),$$

$$\left(X|_{(\Psi^2(X))^c}, \Psi^2(X), \left\{X|_{b_i^2}\right\}_{i\in\mathbb{N}}\right) \stackrel{d}{=} \left(X|_{(\Psi^2(X))^c}, \Psi^2(X), \left\{\delta_{V_i^{21}+c_i^2} + \delta_{V_i^{22}+c_i^2}\right\}_{i\in\mathbb{N}}\right),$$

and

$$\Big(X|_{(\Psi^{1,2}(X))^c}, \Psi^1(X), \Psi^2(X), \Big\{X|_{b_i^1}\Big\}_{i\in\mathbb{N}}, \Big\{X|_{b_i^2}\Big\}_{i\in\mathbb{N}}\Big)$$
$$\stackrel{d}{=} \Big(X|_{(\Psi^{1,2}(X))^c}, \Psi^1(X), \Psi^2(X), \Big\{\delta_{V_i^1+c_i^1}\Big\}_{i\in\mathbb{N}}, \Big\{\delta_{V_i^{21}+c_i^2} + \delta_{V_i^{22}+c_i^2}\Big\}_{i\in\mathbb{N}}\Big).$$

Proof. This follows from Proposition 16 and Lemma 7. □

Proof of Lemma 25. The proof of property (a) follows immediately from the definition of an encoding function, property (b) of a selection rule, and Lemma 24. We now focus our attention on property (b). From the definition of h and the fact that the g_i satisfy the conditions of Lemma 5, it suffices to verify the condition for the simplified encoding function h'.

Let $\mathcal{V}_i^1$ be the unique X-point in each b_i^1. From Corollary 26, we have that $\{\mathcal{V}_i^1 - c_i^1\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. $\mathrm{U}[\bar{B}(\mathbf{0}, R)]$ random variables that is independent of $(X|_{(\Psi^1(X))^c}, \Psi^1(X))$. Similarly, let $\mathcal{V}_i^2$ be the set of unordered X-points of each b_i^2. From Corollary 26, we have that $\{\mathcal{V}_i^2 - c_i^2\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. pairs of unordered $\mathrm{U}[(\bar{B}(\mathbf{0}, R))]$ random variables that is independent of $(X|_{(\Psi^2(X))^c}, \Psi^2(X))$. By the definition of h' and Lemmas 24 and 9, the result follows immediately. □

We turn now to the task of distributing randomness. A natural approach is to have each non-special globe request randomness from the closest available special globe (where distances are measured between the centers of the globes). However, we do not know much about the process of globe-centers. In particular, it is not immediately obvious that it has distinct inter-point distances P-a.s. To avoid this problem, we shall make use of some of the other properties of seeds. Recall that if x is a pre-seed, we call $A(x; R+80, R+90+d)$ the halo. If x is a seed, then we shall also call $A(x; R+80, R+90+d)$ the ***halo***. We shall associate to every globe a point in its halo in an equivariant way.

Tags. Let Ψ be a selection rule from Section 6 and let the inter-point distances of $\mu \in \mathbb{M}$ be distinct. For each globe under μ, we choose a point in its halo in the following isometry-equivariant way. First note that the halo contains more than three μ-points. Take the two mutually closest points in the halo, then choose the one of this pair that is closest to the other points in the halo. We call this point the ***tag*** of the globe. We note that by Lemma 14, part (b), tags are well defined and exist for every globe P-a.s.

For completeness, if the inter-point distances in the halo are not distinct, we take the tag to be the center of the globe. Let $t = t_\Psi : \mathbb{M} \times \mathfrak{F} \to \mathbb{R}^d \cup \{\infty\}$ be the measurable function defined as follows:

$$t_\Psi(\mu, b) \quad := \quad \begin{cases} \text{the tag of } b & \text{if } \mu \in \mathbb{M},\ b \in \text{Globes}[\Psi(\mu)], \\ x & \text{if } \mu \in \mathbb{M},\ b = \{x\}\,,\ x \in [\mu|_{\Psi(\mu)^c}], \\ \infty & \text{otherwise.} \end{cases} \tag{29}$$

We call t_Ψ the ***tagging function*** for the selection rule Ψ.

Lemma 27. *Let $d \geq 1$ and $R > 0$. Let $\Psi = \Psi_R$ be the selection rule from Section 6. The tagging function $t = t_\Psi : \mathbb{M} \times \mathfrak{F} \to \mathbb{R}^d \cup \{\infty\}$ has the following properties.*

1. *The map t depends only on $(\Psi(\mu), \mu\big|_{\Psi(\mu)^c})$; that is, for all $\mu, \mu' \in \mathbb{M}$ if $(\Psi(\mu), \mu\big|_{\Psi(\mu)^c}) = (\Psi(\mu'), \mu'\big|_{\Psi(\mu')^c})$, then $t(\mu, \cdot) = t(\mu', \cdot)$.*

2. *The map t is isometry-equivariant; that is, for all isometries θ of $\mathbb{R}^d$ and for all $(\mu, b) \in \mathbb{M} \times \mathfrak{F}$, $\theta(t(\mu, b)) = t(\theta(\mu), \theta(b))$. Here we take $\theta(\infty) = \infty$.*

Proof. The result follows immediately from the definition of the tagging function. □

Partners and Ranks. Let Ψ be a selection rule from Section 6. We shall now measure distances between globes, as well as distances between globes and μ-points, via the distances between their tags. Let the inter-point distances of $\mu \in \mathbb{M}$ be distinct and also assume that $\text{Globes}^1[\Psi(\mu)]$ and $\text{Globes}^2[\Psi(\mu)]$ are both non-empty. For each globe $b \in \text{Globes}[\Psi(\mu)]$ we call its closest one-special globe its ***one-partner*** (if unique), and its closest two-special globe its ***two-partner*** (if unique). Similarly, for each $x \in [\mu|_{\Psi(\mu)^c}]$ we call its closest one-special globe its ***one-partner*** and its closest two-special globe its ***two-partner***. Suppose that a globe b has a special globe $B \in \text{Globes}^{1,2}[\Psi(\mu)]$ as a partner; then B assigns the number $2n$ to b if there are exactly n globes with B as partner that are closer to B than b. We call the number that b is assigned by its one-partner its ***one-rank*** and the number that b is assigned by its two-partner its ***two-rank***. Similarly, a special globe $B \in \text{Globes}^{1,2}[\Psi(\mu)]$ assigns the number $2n + 1$ to x if it is a partner of $x \in [\mu|_{\Psi(\mu)^c}]$ and there are exactly n partners in $[\mu]$

that are closer to B than x; we also call the number that x is assigned its ***one-rank*** or ***two-rank*** depending on whether it is assigned by its one- or two-partner. Let $\mathbb{M}'$ be the set of point measures of $\mathbb{M}$ that have both one- and two-special globes and have distinct inter-point distances. We define $p = p_\Psi : \mathbb{M} \times (\mathfrak{F} \cup \mathbb{R}^d) \to \mathfrak{F} \times \mathfrak{F}$ as follows:

$$p_\Psi(\mu, b) := (\text{one-partner of } b, \text{two-partner of } b)$$

if $\mu \in \mathbb{M}'$ and $b \in \text{Globes}[\Psi(\mu)] \cup [\mu|_{\Psi(\mu)^c}]$ and $p_\Psi(\mu, b) := (b, b)$ otherwise. We also define $r = r_\Psi : \mathbb{M} \times (\mathfrak{F} \cup \mathbb{R}^d) \to \mathbb{N} \times \mathbb{N}$ as follows:

$$r_\Psi(\mu, b) := (\text{one-rank of } b, \text{two-rank of } b)$$

if $\mu \in \mathbb{M}'$ and $b \in \text{Globes}[\Psi(\mu)] \cup [\mu|_{\Psi(\mu)^c}]$ and $r_\Psi(\mu, b) := (0, 0)$ otherwise. We call p_Ψ the ***partner function*** for the selection rule Ψ and we call r_Ψ the ***rank function*** for Ψ. Also let

$$\chi(\mu) := (\Psi(\mu), \Psi^1(\mu), \Psi^2(\mu), \mu|_{\Psi(\mu)^c})$$

for all $\mu \in \mathbb{M}$.

Lemma 28. *Let $d \geq 1$ and $R > 0$. Let Ψ be the selection rule from Section 6. The partner and rank functions $p = p_\Psi$ and $r = r_\Psi$ have the following properties.*

1. *The maps p, r depend only on $\chi(\mu)$; that is, for all $\mu, \mu' \in \mathbb{M}$ if $\chi(\mu) = \chi(\mu')$, then $p(\mu, \cdot) = p(\mu', \cdot)$ and $r(\mu, \cdot) = r(\mu', \cdot)$*
2. *The map p is isometry-equivariant; that is, for all isometries θ of $\mathbb{R}^d$ and for all $(\mu, b) \in \mathbb{M} \times \mathfrak{F}$, $\theta(p(\mu, b)) = p(\theta(\mu), \theta(b))$.*
3. *The map r is isometry-invariant; that is, for all isometries θ of $\mathbb{R}^d$ and for all $(\mu, b) \in \mathbb{M} \times \mathfrak{F}$, $r(\mu, b) = r(\theta(\mu), \theta(b))$.*

Proof. The result follows immediately from the definitions of the partner and rank functions and Lemma 27. □

Assignment functions. We shall now combine the encoding, partner and rank functions to obtain an assignment function. Let Ψ be a selection rule from Section 6. Define $\boldsymbol{U} = \boldsymbol{U}_\Psi : \mathbb{M} \times (\mathfrak{F} \cup \mathbb{R}^d) \to [0, 1]$ as follows. Let $h = h_\Psi$, $p = p_\Psi$, and $r = r_\Psi$ be the encoding, partner and rank functions. Recall that $h : \mathbb{M} \times \mathfrak{F} \to [0, 1]^{\mathbb{N}}$. For all $(\mu, b) \in \mathbb{M} \times (\mathfrak{F} \cup \mathbb{R}^d)$, let

$$\boldsymbol{U}(\mu, b) := h(\mu, p(\mu, b)_1)_{r(\mu,b)_1} \ \oplus \ h(\mu, p(\mu, b)_2)_{r(\mu,b)_2} \,. \tag{30}$$

Proof of Lemma 23. The isometry-invariance of $\boldsymbol{U}$ follows immediately from the definition of $\boldsymbol{U}$ and Lemmas 25 and 28. Let X be a Poisson point process in $\mathbb{R}^d$. Let $\{\kappa_i\}_{i\in\mathbb{N}} := \text{Globes}[\Psi(X)] \cup [X|_{\Psi(X)^c}]$, $\{b_i^1\}_{i\in\mathbb{N}} := \text{Globes}^1[\Psi(X)]$ and let $\{b_i^2\}_{i\in\mathbb{N}} := \text{Globes}^2[\Psi(X)]$, where we have ordered the sets using the radial ordering. Let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. $\mathrm{U}[0,1]$ random variables independent of X. Let $\{U_i'\}_{i\in\mathbb{N}}$ be an i.i.d. sequence (independent of X), where U_1' is a sequence of i.i.d. $\mathrm{U}[0,1]$ random variables. From Lemma 28, $p(X,\cdot)$ and $r(X,\cdot)$ depend only on $\chi(X)$. It is clear that both $\chi(X)$ and $h(X,b_i^2)$ depend only on $(X|_{(\Psi^1(X))^c}, \Psi^1(X))$, so that by Lemma 25

$$\Big(X|_{(\Psi^1(X))^c}, \Psi^1(X), \chi(X), \big\{h(X,b_i^1)\big\}_{i\in\mathbb{N}}, \big\{h(X,b_i^2)\big\}_{i\in\mathbb{N}}\Big) \stackrel{d}{=} \Big(X|_{(\Psi^1(X))^c}, \Psi^1(X), \chi(X), \{U_i'\}_{i\in\mathbb{N}}, \big\{h(X,b_i^2)\big\}_{i\in\mathbb{N}}\Big). \tag{31}$$

From the definition of the assignment function, it is clear that $\boldsymbol{U}(X,\cdot)$ depends only on

$$\big(\chi(X), \big\{h(X,b_i^1)\big\}_{i\in\mathbb{N}}, \big\{h(X,b_i^2)\big\}_{i\in\mathbb{N}}\big).$$

It is also easy to see that $\chi(X)$ depends only on $(X|_{(\Psi^{1,2}(X))^c}, \Psi^1(X), \Psi^2(X))$. Thus from the definition of the assignment function, (31) and Lemma 9, it follows that

$$\Big(X|_{(\Psi^1(X))^c}, \Psi^1(X), \{\boldsymbol{U}(X,\kappa_i)\}_{i\in\mathbb{N}}\Big) \stackrel{d}{=} \Big(X|_{(\Psi^1(X))^c}, \Psi^1(X), \{U_i\}_{i\in\mathbb{N}}\Big).$$

Similarly, we have that

$$\Big(X|_{(\Psi^2(X))^c}, \Psi^2(X), \chi(X), \big\{h(X,b_i^1)\big\}_{i\in\mathbb{N}}, \big\{h(X,b_i^2)\big\}_{i\in\mathbb{N}}\Big) \stackrel{d}{=} \Big(X|_{(\Psi^2(X))^c}, \Psi^2(X), \chi(X), \big\{h(X,b_i^1)\big\}_{i\in\mathbb{N}}, \{U_i'\}_{i\in\mathbb{N}}\Big), \tag{32}$$

from which it follows that

$$\Big(X|_{(\Psi^2(X))^c}, \Psi^2(X), \{\boldsymbol{U}(X,\kappa_i)\}_{i\in\mathbb{N}}\Big) \stackrel{d}{=} \Big(X|_{(\Psi^2(X))^c}, \Psi^2(X), \{U_i\}_{i\in\mathbb{N}}\Big). \quad \square$$

10 Proof of Theorem 2

In this section, we shall show how the tools used to prove Theorem 1 can be adapted to prove Theorem 2. As a first step we prove a translation-equivariant version of Theorem 2. That is, given λ, λ', we define a translation-equivariant map $\Phi' : \mathbb{M} \to \mathbb{M}$ such that if X is a Poisson process on $\mathbb{R}^d$ of

intensity λ, then $\Phi'(X)$ is a Poisson process of intensity λ'. By modifying the map Φ' we shall obtain a map Υ that is isometry-equivariant and satisfies the conditions of Theorem 2. We need some preliminary definitions before we can give the definition of Φ'.

Voronoi Cells. The Voronoi tessellation of a simple point measure $\mu \in \mathbb{M}$ is a partition of $\mathbb{R}^d$ defined in the following way. The ***Voronoi cell*** of a point $x \in [\mu]$ is the set of all points $y \in \mathbb{R}^d$ such that $\|x-y\| < \|z-y\|$ for all $z \in [\mu] \setminus \{x\}$. The ***unclaimed*** points are the points that do not belong to a cell. We define the ***Voronoi tessellation*** $\mathcal{V}(\mu)$ to be the set of all Voronoi cells along with the set of unclaimed points. Note that if μ is locally finite and not identically zero, then the set of unclaimed points has zero Lebesgue measure. Note that the Voronoi tessellation is clearly isometry-equivariant; that is, for any isometry θ of $\mathbb{R}^d$ we have $\mathcal{V}(\theta\mu) = \theta\mathcal{V}(\mu) := \{\theta\upsilon : \upsilon \in \mathcal{V}(\mu)\}$.

For each $A \in \mathcal{B}$ with positive finite Lebesgue measure, let c_A be its center of mass. Let Ψ be a selection rule and define $\boldsymbol{c} = \boldsymbol{c}_\Psi : \mathbb{M} \to \mathbb{M}$ via

$$\boldsymbol{c}(\mu) := \sum_{b \in \mathrm{Globes}[\Psi(\mu)]} \delta_{c_b} \,. \tag{33}$$

Note that $\boldsymbol{c}$ is also isometry-equivariant. The map Φ' will be defined by placing independent Poisson point processes in each Voronoi cell of $\mathcal{V}(\boldsymbol{c}(\mu))$. Let θ_y be the isometry of $\mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$, we have $\theta_y(x) = x + y$. We define Φ' in the following way. Let Ψ be the R-selection rule from Section 6, where R will be chosen later. Let ϕ^{P} be the collection of mappings from Lemma 6. Let $\boldsymbol{U}$ be the assignment function from Lemma 23. The map $\Phi' = \Phi'_{\lambda'}$ is defined via

$$\Phi'(\mu) := \sum_{\upsilon \in \mathcal{V}(\boldsymbol{c}(\mu))} \sum_{b \in \mathrm{Globes}[\Psi(\mu)]} \mathbf{1}_{[b \subset \upsilon]} \theta_{c_\upsilon}\big(\phi^{\mathrm{P}}_{(\theta^{-1}_{c_\upsilon}(\upsilon), \lambda')}(\boldsymbol{U}(\mu, b))\big) \,. \tag{34}$$

Proposition 29. *The map Φ' has the following properties.*

(a) The map Φ' is translation-equivariant.

(b) If X and Y are Poisson processes on $\mathbb{R}^d$ with intensities $\lambda, \lambda' > 0$, then $\Phi'(X) \stackrel{d}{=} Y$.

Proof. Part (a) follows from the fact that the assignment function is isometry-invariant and that the selection rule, Voronoi tessellation and the map $\boldsymbol{c}$ are

all isometry-equivariant. From the definition of Φ', one can verify that it is translation-equivariant since any two translations of $\mathbb{R}^d$ commute with each other. However, since translations and reflections do not necessarily commute, we have only translation-equivariance. Part (b) follows from Lemma 6 and Lemma 23 once we note that the Voronoi tessellation and the centers of the globes and Voronoi cells depend only on $\Psi(X)$. $\square$

The following example elaborates on the difficulty of defining an isometry-equivariant version of Φ'.

Example 3. *Let $\lambda > 0$. Let $\mathcal{B}^* \subset \mathcal{B}$ be the set of Borel sets with positive finite Lebesgue measure. There does not exist a family of measurable functions ϕ^{p} such that for each $A \in \mathcal{B}^*$, $\phi^{\mathrm{p}}_A : [0,1] \to \mathbb{M}$ has the following properties.*

1. *If U is a $\mathrm{U}[0,1]$ variable, then $\phi^{\mathrm{p}}_A(U)$ is Poisson point process on A with intensity λ'.*

2. *The map ϕ^{p} is isometry-equivariant; that is, for all isometries θ of $\mathbb{R}^d$, $\phi^{\mathrm{p}}_{\theta A}(U) = \theta\phi^{\mathrm{p}}_A(U)$.*

Proof. Towards a contradiction, let ϕ^{p} satisfy the above properties. For each $x \in \mathbb{R}^d$, let x_i be the ith coordinate. Consider $A := B(\mathbf{0}, 1)$, the unit ball centered at the origin, and let $A' := \{x \in A : x_1 > 0\}$. Let θ be the reflection of the first coordinate; that is, if $y = (y_1, \ldots, y_d)$ for some $y_i \in \mathbb{R}$, then $\theta(y) = (-y_1, y_2, \ldots, y_d)$. Let U be a $\mathrm{U}[0,1]$ random variable. The event $E := \{\phi^{\mathrm{p}}_A(U)(A) = \phi^{\mathrm{p}}_A(U)(A') = 1\}$ occurs with non-zero probability. However, $\theta(A) = A$, so that whenever E occurs, $\phi^{\mathrm{p}}_{\theta A}(U) \neq \theta\phi^{\mathrm{p}}_A(U)$. $\square$

Note that in the proof of Example 3, the counterexample used a set A that is invariant under rotations and reflections. One would guess that the Voronoi cells of a random process such as the centers of the special globes should lack such symmetries. However, rather than dealing with the symmetries of the Voronoi cells, we proceed as follows.

Let X be a Poisson process on $\mathbb{R}^d$ with positive intensity and let Ψ be a selection rule from Section 6. Let $b \in \mathrm{Globes}[\Psi(X)]$ and for simplicity assume that the center is at the origin. From the definition of a globe, there will always be at least d points in the halo of a globe. We shall choose d points from the halo and use them to associate an isometry to the globe. Given d points $\{x_1, \ldots, x_d\}$ in the halo of b, we shall define an isometry θ with the following properties.

1. We have $\theta(\mathbf{0}) = \mathbf{0} \in \mathbb{R}^d$.
2. For all i, j such that $1 \leq i < j \leq d$, we have $\theta(x_i)_j = 0 \in \mathbb{R}$; that is, the jth coordinate of $\theta(x_i) \in \mathbb{R}^d$ is zero for $j > i$.
3. For all i such that $1 \leq i \leq d$, we have $\theta(x_i)_i > 0$.

Selecting d points from the halo of a globe is an easy extension of the idea of a tag of a globe. Also to prove that such an isometry exists and is unique, we appeal to the tools of linear algebra, in particular the QR factorization lemma.

Notations and Conventions. To use the tools of linear algebra, it will be convenient to identify elements of $\mathbb{R}^d$ with column vectors; that is, $\mathbb{R}^d = \mathbb{R}^{d\times 1}$. Given an isometry θ of $\mathbb{R}^d$ and a matrix $A \in \mathbb{R}^{d\times d}$, we let $\theta(A) \in \mathbb{R}^{d\times d}$ be the matrix obtained by applying θ to each of the columns of A. We also denote the identity matrix by $I \in \mathbb{R}^{d\times d}$.

d-Tags. Let Ψ be a selection rule from Section 6 and let the inter-point distances of $\mu \in \mathbb{M}$ be distinct. The ***d-tag*** of a globe $b \in \text{Globes}[\Psi(\mu)]$ is a matrix $A \in \mathbb{R}^{d\times d}$ defined inductively as follows. The first column of the matrix is the tag of b. Given that the $(i-1)^{\text{th}}$ column is already defined, the ith column is the μ-point in the halo of b that is closest to the $(i-1)^{\text{th}}$ column and is not equal to any of the first $i-1$ columns. For completeness, if the inter-point distances in the halo are not distinct, we take the d-tag to be the matrix in $\mathbb{R}^{d\times d}$ where each column vector is the center of the globe. Let $\bar{t} = \bar{t}_\Psi : \mathbb{M} \times \mathfrak{F} \to \mathbb{R}^{d\times d} \cup \{\infty\}$ be the measurable function defined as follows:

$$\bar{t}_\Psi(\mu, b) \quad := \quad \begin{cases} \text{the } d\text{-tag of } b & \text{if } \mu \in \mathbb{M} \text{ and } b \in \text{Globes}[\Psi(\mu)], \\ \infty & \text{otherwise.} \end{cases} \tag{35}$$

We call $\bar{t}_\Psi$ the ***d-tagging function*** for the selection rule Ψ.

Lemma 30. *Let $d \geq 1$ and $R > 0$. Let $\Psi = \Psi_R$ be the selection rule from Section 6. The d-tagging function $\bar{t} = \bar{t}_\Psi : \mathbb{M} \times \mathfrak{F} \to \mathbb{R}^{d\times d} \cup \{\infty\}$ has the following properties.*

1. *The map $\bar{t}$ depends only on $(\Psi(\mu), \mu\big|_{\Psi(\mu)^c})$; that is, for all $\mu, \mu' \in \mathbb{M}$, if $(\Psi(\mu), \mu\big|_{\Psi(\mu)^c}) = (\Psi(\mu'), \mu'\big|_{\Psi(\mu')^c})$, then $\bar{t}(\mu, \cdot) = \bar{t}(\mu', \cdot)$.*

2. *The map $\bar{t}$ is isometry-equivariant; that is, for all isometries θ of $\mathbb{R}^d$ and for all $(\mu, b) \in \mathbb{M} \times \mathfrak{F}$, we have $\theta(\bar{t}(\mu, b)) = \bar{t}(\theta(\mu), \theta(b))$. We take $\theta(\infty) = \infty$.*

Proof. The result follows immediately from the definition of the d-tagging function. □

We note that the d-tag of a globe is almost always a non-singular matrix by Lemma 14 (c). The following lemma allows us to associate an isometry to each globe and its d-tag. Recall that every isometry θ of $\mathbb{R}^d$ that fixes the origin can be identified with an unique orthogonal matrix $Q \in \mathbb{R}^{d\times d}$; that is, there is an unique matrix Q such that $QQ^T = Q^TQ = I \in \mathbb{R}^{d\times d}$ and $Qx = \theta(x)$ for all $x \in \mathbb{R}^d = \mathbb{R}^{d\times 1}$. For background, see [20, Ch. 1].

Lemma 31 (QR factorization)**.** *For all $d \geq 1$, if $A \in \mathbb{R}^{d\times d}$ is a square matrix, then there exists an orthogonal matrix $Q \in \mathbb{R}^{d\times d}$ and an upper triangular matrix $\Delta \in \mathbb{R}^{d\times d}$ such that $A = Q\Delta$. Furthermore, if A is non-singular, then the factorization is unique if we require the diagonal entries of Δ to be positive.*

For a proof, see, for example, [9, 2.6].

Upper triangular matrices and fixing isometries. Let Ψ be a selection rule from Section 6 and let $b \in \text{Globes}[\Psi(\mu)]$. The ***upper triangular matrix for*** b is the matrix $\Delta \in \mathbb{R}^{d\times d}$ defined as follows. Let c_b be the center of the globe b. Let $A' \in \mathbb{R}^{d\times d}$ be the d-tag for the globe b. Let $A := A' - c_bI$. If A is singular, then we take $\Delta = \mathbf{0} \in \mathbb{R}^{d\times d}$. Otherwise, by Lemma 31, there exists a unique factorization such that $A = Q\Delta$, where $\Delta \in \mathbb{R}^{d\times d}$ is an orthogonal matrix and $\Delta \in \mathbb{R}^{d\times d}$ is an upper triangular matrix such that all its diagonal entries are positive. When A is non-singular, we say that the unique isometry σ such that $\sigma(c_b) = \mathbf{0} \in \mathbb{R}^d$ and $\sigma(A') = \Delta$ is the ***fixing isometry*** for the globe b.

Let $\mathbf{\Delta} = \mathbf{\Delta}_\Psi : \mathbb{M} \times \mathfrak{F} \to \mathbb{R}^{d\times d}$ be the measurable function defined as follows:

$$\mathbf{\Delta}_\Psi(\mu, b) := \text{the upper triangular matrix for } b$$

if $\mu \in \mathbb{M}$ and $b \in \text{Globes}[\Psi(\mu)]$, while $\mathbf{\Delta}_\Psi(\mu, b) := I \in \mathbb{R}^{d\times d}$ otherwise. Let $\mathbf{0} \in (\mathbb{R}^d)^{\mathbb{R}^d}$ be the function that sends every element of $\mathbb{R}^d$ to $\mathbf{0} \in \mathbb{R}^d$. The

fixing isometry function $\boldsymbol{\sigma} = \boldsymbol{\sigma}_\Psi : \mathbb{M} \times \mathfrak{F} \to (\mathbb{R}^d)^{\mathbb{R}^d}$ for the selection rule Ψ is defined as follows:

$$\boldsymbol{\sigma}_\Psi(\mu, b) := \text{the fixing isometry for the globe } b$$

if $\mu \in \mathbb{M}$, $b \in \text{Globes}[\Psi(\mu)]$, and the d-tag of b is non-singular, while $\boldsymbol{\sigma}_\Psi(\mu, b) := \mathbf{0} \in (\mathbb{R}^d)^{\mathbb{R}^d}$ otherwise.

Lemma 32. *Let $d \geq 1$ and $R > 0$. Let $\Psi = \Psi_R$ be the selection rule from Section 6. The map $\boldsymbol{\Delta} = \boldsymbol{\Delta}_\Psi : \mathbb{M} \times \mathfrak{F} \to \mathbb{R}^{d \times d}$ and the fixing isometry function $\boldsymbol{\sigma} = \boldsymbol{\sigma}_\Psi : \mathbb{M} \times \mathfrak{F} \to (\mathbb{R}^d)^{\mathbb{R}^d}$ have the following properties.*

1. *The maps $\boldsymbol{\Delta}$ and $\boldsymbol{\sigma}$ depend only on $(\Psi(\mu), \mu\big|_{\Psi(\mu)^c})$; that is, for all $\mu, \mu' \in \mathbb{M}$ if $(\Psi(\mu), \mu\big|_{\Psi(\mu)^c}) = (\Psi(\mu'), \mu'\big|_{\Psi(\mu')^c})$, then $\boldsymbol{\Delta}(\mu, \cdot) = \boldsymbol{\Delta}(\mu', \cdot)$ and $\boldsymbol{\sigma}(\mu, \cdot) = \boldsymbol{\sigma}(\mu', \cdot)$.*

2. *The map $\boldsymbol{\Delta}$ is isometry-invariant; that is, for all isometries θ of $\mathbb{R}^d$ and for all $(\mu, b) \in \mathbb{M} \times \mathfrak{F}$, we have $\boldsymbol{\Delta}(\mu, b) = \boldsymbol{\Delta}(\theta(\mu), \theta(b))$.*

Proof. The first property follows immediately from the definitions of the maps and Lemma 30. We prove the second property in the following way. Let θ be an isometry of $\mathbb{R}^d$. Let $A' \in \mathbb{R}^{d \times d}$ be a non-singular matrix, let $a' \in \mathbb{R}^d$, and set $A := A' - a'I$, where I is the identity matrix. Let $A = Q\Delta$ be the unique QR factorization of A, where all the diagonal entries of Δ are positive. From the definition of the upper triangular matrix for a globe, it suffices to show that for some orthogonal matrix Q'', we have

$$\theta(A') - \theta(a')I = Q''\Delta\,.$$

Note that there exists an orthogonal matrix Q' and $c \in \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d = \mathbb{R}^{d \times 1}$, we have $\theta(x) = Q'x + c$. Observe that

$$\begin{aligned}
\theta(A') - \theta(a')I &= Q'A' + cI - (Q'a' + cI) \\
&= Q'(A' - a'I) \\
&= Q'A \\
&= (Q'Q)\Delta\,.
\end{aligned}$$

□

We are now ready to give the definition of the mapping that satisfies the conditions of Theorem 2. Let X and Y be Poisson processes on $\mathbb{R}^d$ with positive intensities $\lambda, \lambda' > 0$. Set $R = 1$ and let Ψ be the R-selection rule

from Section 6 with $R = 1$. Let $\boldsymbol{\sigma} : \mathbb{M} \times \mathfrak{F} \to (\mathbb{R}^d)^{\mathbb{R}^d}$ be the fixing isometry function for Ψ, let ϕ^{P} be a collection of functions from Lemma 6, and let $\boldsymbol{U}$ be the assignment function from Lemma 23. Define $\Upsilon = \Upsilon_{\lambda'} : \mathbb{M} \to \mathbb{M}$ as

$$\Upsilon(\mu) := \sum_{\upsilon \in \mathcal{V}(\boldsymbol{c}(\mu))} \sum_{b \in \mathrm{Globes}[\Psi(\mu)]} \mathbf{1}_{[b \subset \upsilon]} \mathbf{1}_{[\boldsymbol{\sigma}(\mu,b) \neq \mathbf{0}]} \boldsymbol{\sigma}(\mu, b)^{-1} \left(\phi^{\mathrm{P}}_{(\boldsymbol{\sigma}(\mu,b)(\upsilon), \lambda')} (\boldsymbol{U}(\mu, b)) \right) \tag{36}$$

for all $\mu \in \mathbb{M}$.

Proof of Theorem 2. From the definition of Υ it is almost immediate that it is isometry-equivariant. It suffices to check the following claim. Let $\upsilon \in \mathcal{B}$, let $b \in \mathrm{Globes}[\Psi(\mu)]$, and let θ be any isometry of $\mathbb{R}^d$. We claim that for all $\mu \in \mathbb{M}$,

$$\boldsymbol{\sigma}(\theta\mu, \theta b)^{-1} \Big(\phi^{\mathrm{P}}_{\boldsymbol{\sigma}(\theta\mu,\theta b)(\theta\upsilon)} \left(\boldsymbol{U}[\theta\mu, \theta b] \right) \Big) = \theta \Big(\boldsymbol{\sigma}(\mu, b)^{-1} \left(\phi^{\mathrm{P}}_{\boldsymbol{\sigma}(\mu,b)(\upsilon))} \left(\boldsymbol{U}[\mu, b] \right) \right) \Big). \tag{37}$$

To check (37), observe that by Lemma 32 and the definition of the fixing isometry function,

$$\boldsymbol{\sigma}(\theta\mu, \theta b) = \boldsymbol{\sigma}(\mu, b) \circ \theta^{-1} .$$

Hence, $\boldsymbol{\sigma}(\theta\mu, \theta b)^{-1} = \theta \circ \boldsymbol{\sigma}(\mu, b)^{-1}$ and $\boldsymbol{\sigma}(\theta\mu, \theta b)(\theta\upsilon) = \boldsymbol{\sigma}(\mu, b)(\upsilon)$. In addition, by Lemma 23 (part (b)), $\boldsymbol{U}(\mu, b) = \boldsymbol{U}(\theta\mu, \theta b)$, whence

$$\phi^{\mathrm{P}}_{\boldsymbol{\sigma}(\theta\mu,\theta b)(\theta\upsilon)} \left(\boldsymbol{U}(\theta\mu, \theta b) \right) = \phi^{\mathrm{P}}_{\boldsymbol{\sigma}(\mu,b)(\upsilon)} \left(\boldsymbol{U}(\mu, b) \right) .$$

Thus, (37) holds.

Let X and Y be Poisson point processes on $\mathbb{R}^d$ with intensities $\lambda, \lambda' > 0$. Again it follows from Lemma 6 and Lemma 23 that $\Upsilon(X) \stackrel{d}{=} Y$. We need only note the following: the Voronoi tessellation and the centers of the globes and Voronoi cells depend only on $\Psi(X)$ (as in the case of Φ' from Proposition 29) and from Lemma 32, the fixing isometry function $\boldsymbol{\sigma}$ also depends only on $(\Psi(X), X|_{\Psi(X)^c})$. □

11 Proof of Theorem 4

In this section, we shall prove Theorem 4 by showing that the map Γ defined in (24) and used to prove Theorem 1 and the map Υ defined in (36) and used to prove Theorem 2 are both strongly finitary. We shall prove the following stronger result from which Theorem 4 follows immediately.

Theorem 33. *Let Γ and Υ be the maps defined in (24) and (36) respectively. There exists a map $T : \mathbb{M} \to \mathbb{N} \cup \{\infty\}$ such that if X is a Poisson point process on $\mathbb{R}^d$ with positive intensity, then $\mathbb{E}T(X)$ is finite and for all $\mu, \mu' \in \mathbb{M}$ such that $T(\mu), T(\mu') < \infty$ and $\mu|_{B(\mathbf{0},T)} = \mu'|_{B(\mathbf{0},T)}$, we have*

$$\Gamma(\mu)|_{B(\mathbf{0},1)} = \Gamma(\mu')|_{B(\mathbf{0},1)} \quad \textit{and} \quad \Upsilon(\mu)|_{B(\mathbf{0},1)} = \Upsilon(\mu')|_{B(\mathbf{0},1)} \,.$$

Let P be the law of X. Since $\mathbb{E}T(X) < \infty$ implies that $T(X)$ is finite P-a.s., Theorem 33 implies that Γ and Υ are both strongly finitary with respect to P.

We shall require the following additional property that the selection rules defined in Section 6 satisfy.

Lemma 34. *Let Ψ_R be a selection rule from Section 6. For any $z \in \mathbb{R}^d$ any $\mu, \mu' \in \mathbb{M}$, if $\bar{B}(z, R)$ is a globe under μ, then whenever $\mu|_{B(z,R+120+d)} = \mu'|_{B(z,R+120+d)}$, we have that $\bar{B}(z, R)$ is also a globe under μ'.*

Lemma 34 is a localized version of property (c) in the definition of a selection rule. We omit the proof of Lemma 34, which uses the definition of pre-seeds and seeds and is similar to that of Lemma 22.

Proof of Theorem 33. Let $\Psi = \Psi_R$ be the R-selection rule from Section 6 that is used to define the map $\Gamma = \Gamma_{(\lambda,\lambda')}$. Recall that we use the R-selection rule with $R = 1$ to define the map Υ. We now work towards a definition of T. Fix $r := 100(R+101+d)$. Let $\{C_i\}_{i\in\mathbb{Z}^d}$ be an indexed partition of $\mathbb{R}^d$ into equal-sized cubes of side length r such that C_i is centered at ir. For all $i \in \mathbb{Z}^d$, let $c_i \subset C_i$ be the ball of radius 1 concentric with the cube C_i and let $E_i \in \mathcal{M}$ be the set of measures such that c_i contains a seed. Because the radius of c_i is 1, it never contains more than one seed. Let X be a Poisson point process on $\mathbb{R}^d$ with positive intensity and law P. It follows from the definition of a seed that $\{\mathbf{1}_{X\in E_i}\}_{i\in\mathbb{Z}^d}$ is a collection of i.i.d. random variables with positive expectation. For each $i \in \mathbb{Z}^d$, let $E_i^1 \subset E_i$ be the set of measures where the globe corresponding to the seed in c_i is one-special and similarly let $E_i^2 \subset E_i$ be the set of measures where the globe corresponding to the seed in c_i is two-special. By Proposition 16, it follows that $\{\mathbf{1}_{X\in E_i^1}\}_{i\in\mathbb{Z}^d}$ and $\{\mathbf{1}_{X\in E_i^2}\}_{i\in\mathbb{Z}^d}$ are collections of i.i.d. random variables with positive expectation. Let

$$\begin{aligned} T_1^1(\mu) \;:=\; \inf\big\{ n \in \mathbb{Z}^+ : \mu \in E^1_{(n,0,\ldots,0)} &\text{ and for some } 0 < k_1 < k_2 < n, \\ &\text{we have } \mu \in E^1_{(k_j,0,\ldots,0)} \text{ for } j = 1,2 \big\} \,. \end{aligned}$$

Also define

$$T_1^2(\mu) \;:=\; \inf\big\{n \in \mathbb{Z}^+ : \mu \in E^2_{(n,0,\dots,0)} \text{ and for some } 0 < k_1 < k_2 < n, \\ \text{we have } \mu \in E^2_{(k_j,0,\dots,0)} \text{ for } j = 1,2\big\}\,.$$

Similarly, define T_i^1 and T_i^2 for all $2 \le i \le d$ by using coordinate i. Clearly, for all $1 \le i \le d$, both $T_i^1(X)$ and $T_i^2(X)$ have finite mean. We set

$$T := 8\sum_{j=1}^{d} r(T_i^1 + T_i^2)\,.$$

We now show that Γ satisfies the required property. Let $\mathbb{M}_T \subset \mathbb{M}$ be the set of point measures such that $\mu \in \mathbb{M}_T$ iff $T(\mu) < \infty$. Observe that since Γ is monotone, to determine $\Gamma(\mu)|_{B(\mathbf{0},1)}$ it suffices to determine which points of $[\mu] \cap B(\mathbf{0},1)$ will be in $[\Gamma(\mu)] \cap B(\mathbf{0},1)$. If $x \in [\mu]$ does not belong to a globe, then whether or not it is deleted depends on the value of $\boldsymbol{U}(\mu,x)$. Recall that $\boldsymbol{U}$ is the assignment function for Ψ. If $x \in [\mu] \cap B(\mathbf{0},1)$ does belong to a globe, then whether or not it is deleted depends on the globe b for which $x \in b$, on $\boldsymbol{U}(\mu,b)$, and on the splitting $\phi_b^{\text{fin}}(\mu|_b, \boldsymbol{U}(\mu,b))$. Let $\boldsymbol{c}$ be defined as in (33), the point process of the centers of the globes. Thus it suffices to show that for all $\mu,\mu' \in \mathbb{M}_T$ such that $\mu|_{B(\mathbf{0},T(\mu))} = \mu'|_{B(\mathbf{0},T(\mu))}$, we have

(a) $\Psi(\mu) \cap B(\mathbf{0},1) = \Psi(\mu') \cap B(\mathbf{0},1)$ and $\boldsymbol{c}\big(\Psi(\mu)\big)\big|_{B(\mathbf{0},1)} = \boldsymbol{c}\big(\Psi(\mu')\big)\big|_{B(\mathbf{0},1)}$,

(b) $\boldsymbol{U}(\mu,x) = \boldsymbol{U}(\mu',x)$ for all $x \in B(\mathbf{0},1)$, and

(c) $\boldsymbol{U}(\mu,\bar{B}(y,R)) = \boldsymbol{U}(\mu',\bar{B}(y,R))$ for all $y \in B(\mathbf{0},1)$.

Write $T = T(\mu)$. Property (a) follows from $\mu|_{B(\mathbf{0},T)} = \mu'|_{B(\mathbf{0},T)}$ and Lemma 34. Property (b) follows from the following observations. If $\bar{B}(z,R)$ is a partner of some $x \in B(\mathbf{0},1)$, then $\|x-z\| \le r\max(T_1^1,T_1^2)$. Thus $B(z,R+120+d) \subset B(\mathbf{0},T)$. In addition, if $y \in [\mu]$ shares a partner with x and has lower one- or two-rank than x, then $\|x-y\| \le 2r\max(T_1^1,T_1^2)$, so that $y \in B(\mathbf{0},T)$. Also note that the tag of a globe $B(z,R)$ is contained in $B(z,R+120+d)$. Hence by Lemma 34, the partners and ranks of x are determined on $B(\mathbf{0},T)$. Thus

$$p(\mu,x) = p(\mu',x) \quad \text{and} \quad r(\mu,x) = r(\mu',x)$$

for all $x \in B(\mathbf{0}, 1)$ and all $\mu, \mu' \in \mathbb{M}_T$ such that $\mu|_{B(\mathbf{0},T)} = \mu'|_{B(\mathbf{0},T)}$, where p, r are the partner and rank functions of Ψ. From the definition of $\boldsymbol{U}$, property (b) follows.

Similarly, if $y \in B(\mathbf{0}, 1)$, $b = \bar{B}(y, R)$ is a globe, and $\bar{B}(z, R)$ is a partner of b, then $B(z, R+120+d) \subset B(\mathbf{0}, T)$. In addition, if $\bar{B}(y, R)$ shares a partner with b and has a lower rank than b, then $B(y, R + 120 + d) \subset B(\mathbf{0}, T)$. Thus property (c) also holds.

The proof that Υ has the required property is similar. Recall that Υ is defined by placing Poisson point processes inside each member of $\mathcal{V}(\boldsymbol{c})$ using the assignment function $\boldsymbol{U}$. Recall that $\mathcal{V}(\mu)$ is the Voronoi tessellation of the point process μ and each Voronoi cell receives the U$[0, 1]$ variable assigned to the globe that is contained in the Voronoi cell. For all $x \in \mathbb{R}^d$, let $v(x, \mu)$ be the member of $\mathcal{V}(\boldsymbol{c}(\mu))$ to which x belongs. From the definition of T and Lemma 34, it follows that for all $x \in B(\mathbf{0}, 1)$ and all $\mu, \mu' \in \mathbb{M}_T$ such that $\mu|_{B(\mathbf{0},T(\mu))} = \mu'|_{B(\mathbf{0},T(\mu))}$, we have $v(x, \mu) = v(x, \mu')$. Moreover, it is not difficult to verify that for each $x \in B(\mathbf{0}, 1)$, if $b \subset v(x, \mu)$ is a globe, then its partners, rank and assignment function are also determined on $B(\mathbf{0}, T(\mu))$. □

12 Open Problems

Question 1, in the introduction, asked whether a homogeneous Poisson point process X on $\mathbb{R}^d$ can be deterministically "thickened" via a factor—that is, whether there exists a deterministic isometry-equivariant map ϕ such that $\phi(X)$ is a homogeneous Poisson process of higher intensity that contains all the original points of X.

We also do not know the answer to the following question, where we drop the requirement of equivariance.

Question 2. *Let $d \geq 1$ and let $\lambda' > \lambda > 0$. Does there exist a deterministic map ϕ such that if X is a homogeneous Poisson point process, then $\phi(X)$ is a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity λ', and such that* all *the points of X are points of $\phi(X)$?*

We can also ask similar questions in the discrete setting of Bernoulli processes. We do not know the answer to following simple question.

Question 3. *Let $X = \{X_i\}_{i \in \mathbb{Z}}$ be a sequence of i.i.d. $\{0, 1\}$-valued random variables with $\mathbb{E}(X_0) = \frac{1}{4}$. Does there exist a deterministic map f such*

that $\{f(X)_i\}_{i\in\mathbb{Z}}$ *is a sequence of i.i.d.* $\{0,1\}$*-valued random variables with* $\mathbb{E}(f(X)_0) = \frac{1}{2}$ *and* $f(X)_i \geq X_i$ *for all* $i \in \mathbb{Z}$*?*

Note that there does not exist a translation-equivariant map ϕ, a factor, that satisfies the condition of Question 3; if ϕ is a factor, then by the Kolmogorov-Sinai theorem, the entropy of $\phi(X)$ can not be greater than the entropy of X. See [19, Chapter 5] for more details. More generally, if $B(p)$ and $B(q)$ are Bernoulli shifts on $\{0, 1, \ldots, d-1\}$, where the entropy of p is less than the entropy of q, one can ask whether there exists a deterministic map ϕ from $B(p)$ to $B(q)$ such that we have $\phi(x)_i \geq x_i$ for all $x \in \{0, 1, \ldots, d-1\}$ and all $i \in \mathbb{Z}$. Also see [2] for more open problems.

Remark. Ori Gurel-Gurevich and Ron Peled have informed us that they have answered Questions 1–3 (with respective answers no, yes, and yes) in a manuscript entitled "Poisson Thickening".

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