

A SHARPER THRESHOLD FOR BOOTSTRAP PERCOLATION IN TWO DIMENSIONS

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ABSTRACT. Two-dimensional bootstrap percolation is a cellular automaton in which sites become ‘infected’ by contact with two or more already infected nearest neighbors. We consider these dynamics, which can be interpreted as a monotone version of the Ising model, on an $n \times n$ square, with sites initially infected independently with probability p . The critical probability p_c is the smallest p for which the probability that the entire square is eventually infected exceeds $1/2$. Holroyd determined the sharp first-order approximation: $p_c \sim \pi^2/(18 \log n)$ as $n \rightarrow \infty$. Here we sharpen this result, proving that the second term in the expansion is $-(\log n)^{-3/2+o(1)}$, and moreover determining it up to a poly($\log \log n$)-factor.

1. INTRODUCTION

Bootstrap percolation is a cellular automaton in which, given a (typically random) initial set of ‘infected’ vertices in a graph G , new vertices are infected at each time step if they have at least r infected neighbours. In this paper we shall study two-neighbour bootstrap percolation on the square grid $[n]^2$. We shall determine the second term of the critical threshold for percolation up to a poly($\log \log n$)-factor, and hence confirm a conjecture of Gravner and Holroyd [23].

We begin by defining the bootstrap process, which was introduced by Chalupa, Leath and Reich [17] in 1979. Given a graph G , let $V(G)$ denote its vertex set, and given $v \in V(G)$, let $N(v)$ denote the neighbourhood of v . Now, given an integer $r \in \mathbb{N}$, and a set of initially infected vertices $A \subset V(G)$, define A_t recursively by $A_0 = A$, and

$$A_{t+1} = A_t \cup \{v \in V(G) : |N(v) \cap A_t| \geq r\}$$

for each $t \geq 0$. We say that the vertices of A_t have been infected by time t . Let $[A] = \bigcup_t A_t$ denote the closure of A under the r -neighbour bootstrap process, and say that the set A *percolates* if the entire vertex set is eventually infected, i.e., if $[A] = V(G)$.

We shall be interested in the case where A is a random subset of $V(G)$ in which each vertex is included independently with probability p . It is clear that the probability of percolation is strictly increasing in p , and so we define the critical probability, $p_c(G, r)$ as follows:

$$p_c(G, r) := \inf \left\{ p : \mathbb{P}(A \text{ percolates in the } r\text{-neighbour process on } G) \geq 1/2 \right\}.$$

Our aim is to give sharp bounds on $p_c(G, r)$.

Bootstrap percolation has been extensively studied, both by mathematicians (see, for example [2, 6, 15, 25, 32]) and by physicists (see the survey [1] and the references therein).

The process may be thought of as a monotone version of the Ising model. We focus on the graph $G = [n]^d$ with vertex set $\{1, \dots, n\}^d$ and an edge between vertices u and v if and only if $\|u - v\|_1 = 1$. Aizenman and Lebowitz [2] determined the asymptotic behaviour of $p_c([n]^d, 2)$ up to multiplicative constants, and Cerf and Cirillo [15] (in the crucial case $d = r = 3$) and Cerf and Manzo [16] proved the corresponding result for all $r \leq d$. The first sharp threshold for bootstrap percolation was proved by Holroyd [25], who showed that

$$p_c([n]^2, 2) = \frac{\pi^2}{18 \log n} + o\left(\frac{1}{\log n}\right). \quad (1)$$

This was the first result of its type, and has prompted a flurry of generalizations. Sharp thresholds have since been determined for $p_c([n]^d, r)$ for all fixed d and r [6, 8], and in high dimensions (i.e., $d = d(n) \rightarrow \infty$ sufficiently fast) when $r = \Theta(d)$ [5], and when $r = 2$ [7]. Some of the techniques from these papers have been used to prove results about the Glauber dynamics of the Ising model [20, 31]. The bootstrap process has also been studied on trees [9, 12, 19], on the random regular graph [10, 28], and on $G_{n,p}$ [29].

In this paper we shall study the two-neighbour bootstrap process on the graph $G = [n]^2$ in more detail. One of the most striking facts about the result (1) stated above is that it contradicted estimates of $\lim_{n \rightarrow \infty} p_c \log n$ given by simulations - in fact, such estimates were out by a factor of more than two. (See, for example, [22] or [24] for a discussion of the reasons behind these discrepancies.) Gravner and Holroyd [22] gave a rigorous (partial) explanation for this phenomenon, by giving the following improvement of (1):

$$p_c([n]^2, 2) \leq \frac{\pi^2}{18 \log n} - \frac{c}{(\log n)^{3/2}},$$

where $c > 0$ is a small constant. In [23], the same authors proved an almost matching lower bound for a simpler model (called ‘local’ bootstrap percolation), and conjectured that the upper bound is essentially sharp for the usual bootstrap process.

Conjecture 1 (Gravner and Holroyd [23]). *For every $\varepsilon > 0$, if n is sufficiently large then*

$$p_c([n]^2, 2) \geq \frac{\pi^2}{18 \log n} - \frac{1}{(\log n)^{3/2-\varepsilon}}.$$

In this paper we shall prove Conjecture 1 in a slightly stronger form. To be precise, we shall prove the following theorem.

Theorem 1. *There exist constants $C > 0$ and $c > 0$ such that*

$$\frac{\pi^2}{18 \log n} - \frac{C(\log \log n)^3}{(\log n)^{3/2}} \leq p_c([n]^2, 2) \leq \frac{\pi^2}{18 \log n} - \frac{c}{(\log n)^{3/2}}$$

for every $n \in \mathbb{N}$.

Note that the upper bound follows from the result of [22] stated above, and so we shall need to prove only the lower bound. We remark that our result corrects predictions (from simulations) of the power of $\log n$ in the second term (see [23] for details).

The proof of Theorem 1 will use many of the tools and techniques of [25], together with some of the ideas of [23], and some new ideas. In particular, we shall bound the probability of percolation by the expected number of ‘good’ and ‘satisfied’ hierarchies (see Lemma 7, below). We will define a hierarchy as in [25] (see Section 3), except that our hierarchies will be much finer, each step being of order $1/\sqrt{p}$, instead of $1/p$. This means that we will have far too many hierarchies; however, almost all of these have many ‘large’ seeds, and we shall show that these contribute a negligible amount to the sum. In order to do so, we shall need a better bound on the probability that a seed is internally spanned than the straightforward bound that sufficed in [25]. Fortunately, the bound we need follows easily from the simple (folklore) fact that a spanning set for a rectangle R must contain no fewer than $\phi(R)/2$ elements, where $\phi(R)$ denotes the semi-perimeter of R (see Lemmas 2 and 3). Surprisingly, it appears that our proof does not extend directly to the “modified” bootstrap percolation model; it is the analogous bound for seeds that is missing in this case (see Section 5 for more information).

We finish this section by making a few definitions which we shall use throughout the proof. First, we say a set S is *spanned* by a set A if $S \subset [A]$, and that S is *internally spanned* by A if $S \subset [A \cap S]$. We write $A \sim \text{Bin}(S, p)$ to indicate that A is a random subset of S , with each element chosen independently with probability p , and write \mathbb{P}_p for the corresponding probability measure. Let $I(S)$ denote the event that S is internally spanned by A . Thus $\mathbb{P}_p(I(S))$ is the probability that the set S is internally spanned by a random set $A \sim \text{Bin}(S, p)$.

Next, define two functions, β and g , by

$$\beta(u) := \frac{u + \sqrt{u(4 - 3u)}}{2} \quad \text{and} \quad g(z) := -\log(\beta(1 - e^{-z})).$$

We remark that β is increasing on $[0, 1]$, and so g is decreasing on $(0, \infty)$, and that $g(z) \leq 2e^{-z}$ when z is large (see Proposition 3 of [6]). Note that $\beta(u) \sim \sqrt{u}$ as $u \rightarrow 0$, and so $g(z) \sim -\log \sqrt{z}$ as $z \rightarrow 0$.

A *rectangle* is a set of the form

$$R = [(a, b), (c, d)] := \{(x, y) : a \leq x \leq c, b \leq y \leq d\} \subset \mathbb{Z}^2,$$

the dimensions of R are $\dim(R) = (c - a + 1, d - b + 1)$, the long and short side-lengths of R are respectively $\text{sh}(R) = \min\{c - a + 1, d - b + 1\}$ and $\text{lg}(R) = \max\{c - a + 1, d - b + 1\}$, and the semi-perimeter of R is $\phi(R) = \text{sh}(R) + \text{lg}(R)$.

We say that a rectangle $R = [(a, b), (c, d)]$ is *crossed from left-to-right* by $A \subset R$ if

$$R \subset [A \cup \{(x, y) \in \mathbb{Z}^2 : x \leq a - 1\}],$$

i.e., if R is spanned by A together with the set of all sites to the left of R . Note that this is equivalent to there being no ‘double gap’ (i.e., no adjacent pair of empty columns) in R , and the final column being occupied.

For each $p \in (0, 1)$, let $q = -\log(1 - p)$, so that $p \sim q$ as $p \rightarrow 0$. To motivate this definition (and the definition of $g(z)$, above), note (from Lemma 8 of [25]) that for any

rectangle R with dimensions (a, b) , if $A \sim \text{Bin}(R, p)$ then

$$\mathbb{P}(A \text{ crosses } R \text{ from left-to-right}) \leq e^{ag(bq)}.$$

We shall use the notation $f(\mathbf{x}) = O(h(\mathbf{x}))$ throughout to mean that there exists an absolute constant $C > 0$, independent of all other variables (unless otherwise stated), such that $f(\mathbf{x}) \leq Ch(\mathbf{x})$ for all $\mathbf{x} = (x_1, \dots, x_k)$. If the constant C depends on some other parameter y , then we shall write $f(\mathbf{x}) = O_y(h(\mathbf{x}))$. Finally, given a directed tree, let $\vec{\Gamma}(v)$ denote the set of out-neighbours of a vertex v .

The rest of the paper is organised as follows. In Section 2 we give an upper bound on the probability that a sufficiently small rectangle (a seed) is internally spanned. In Section 3 we recall from [25] the notion of a hierarchy, which is fundamental to the proof of Theorem 1, together with some important lemmas from [23] and [25]. In Section 4 we prove Theorem 1, and in Section 5 we mention some open questions.

2. A LEMMA ON SEEDS

In this section we shall prove the following lemma, which bounds the probability that a small rectangle is internally spanned. Recall that $q = -\log(1 - p)$.

Lemma 2. *There exists $c > 0$ such that, for any $p > 0$ and any rectangle R with $\dim(R) = (a, b)$ and $a \leq b$, if $ap \leq c$ then*

$$\mathbb{P}_p(I(R)) \leq 3^{\phi(R)} \exp\left(-\phi(R)g(aq)\right).$$

We begin by recalling a lovely and well-known exercise for high school students (see [13] or [33], for example). Lemma 2 follows from it almost immediately.

Lemma 3. *If $A \subset R$ percolates then $|A| \geq \phi(R)/2$.*

We also make a simple observation.

Observation 4. *If $z > 0$ is sufficiently small then*

$$\log(1/\sqrt{z}) - \sqrt{z} \leq g(z) \leq \log(1/\sqrt{z}) + z.$$

Proof. We use the estimates $z - z^2 \leq 1 - e^{-z} \leq z$, and $\sqrt{u} \leq \beta(u) \leq \sqrt{u} + u$, which are valid for small z and u . It follows that

$$g(z) \geq -\log \beta(z) \geq -\log(\sqrt{z} + z) = -\log \sqrt{z} - \log(1 + \sqrt{z}) \geq -\log \sqrt{z} - \sqrt{z}.$$

The proof of the upper bound is similar. □

We can now easily deduce Lemma 2.

Proof of Lemma 2. Let $m = |A \cap R|$. By Lemma 3, if A internally spans R then $m \geq (a + b)/2$. There are at most $\binom{ab}{m}$ ways to choose the set A , given m . Thus

$$\mathbb{P}_p(I(R)) \leq \sum_{m \geq (a+b)/2} \binom{ab}{m} p^m \leq (6aq)^{(a+b)/2},$$

since $\varepsilon := aq$ is sufficiently small, and $p \sim q$. But $\log(1/\sqrt{aq}) \leq g(aq) + \sqrt{aq}$, by Observation 4, so

$$(aq)^{(a+b)/2} \leq \exp\left(- (a+b)g(aq) + (a+b)\sqrt{aq}\right).$$

The result now follows, since $aq = \varepsilon$, and $\sqrt{6}e^{\sqrt{\varepsilon}} < 3$ if ε is sufficiently small. \square

3. HIERARCHIES

In this section we shall recall some important definitions and lemmas from [23] and [25]; for the proofs, we refer the reader to those papers. In particular, we define a hierarchy as in Section 9 of [25].

Definition. A *hierarchy* \mathcal{H} for a rectangle $R \subset [n]^2$ is an oriented rooted tree $G_{\mathcal{H}}$, with all edges oriented away from the root (‘downwards’), together with a collection of rectangles $(R_u \subset [n]^2 : u \in V(G_{\mathcal{H}}))$, one for each vertex of $G_{\mathcal{H}}$, satisfying the following criteria.

- (a) The root of $G_{\mathcal{H}}$ corresponds to R .
- (b) Each vertex has at most 2 neighbours below it.
- (c) If $u \rightarrow v$ in $G_{\mathcal{H}}$ then $R_u \supset R_v$.
- (d) If $\vec{\Gamma}(u) = \{v, w\}$ then $[R_v \cup R_w] = R_u$.

A vertex u with $\vec{\Gamma}(u) = \emptyset$ is called a *seed*. Given two rectangles $S \subset R$, we write $D(S, R)$ for the event (depending on the set $A \subset R$) that

$$R = [(A \cup S) \cap R],$$

i.e., the event that R is internally spanned by $A \cup S$.

We say a hierarchy *occurs* (or is *satisfied* by a set $A \subset R$) if the following events all occur *disjointly*.

- (e) For each seed u : R_u is internally spanned by A .
- (f) For each pair (u, v) satisfying $\vec{\Gamma}(u) = \{v\}$: $D(R_v, R_u)$.

Given two rectangles $S \subset R$, with dimensions (a_1, a_2) and (b_1, b_2) respectively, define

$$d_j(S, R) := \frac{b_j - a_j}{b_j}$$

for $j = 1, 2$, and let $d(S, R) = \max\{d_1(S, R), d_2(S, R)\}$.

The following definition is slightly different to that in [25], and is motivated by the method of [23] (see also Lemma 9 below). This definition is necessary because in order to prove a sharper result, we need to take a finer hierarchy. In our application we shall take $T = \sqrt{q}$ and $Z = \log^3(1/q)/\sqrt{q}$.

Definition. A hierarchy is *good* for $(T, Z) \in \mathbb{R}^2$ if it satisfies the following.

- (g) If $\vec{\Gamma}(u) = \{v\}$ and $|\vec{\Gamma}(v)| = 1$ then $T \leq d(R_v, R_u) \leq 2T$.
- (h) If $\vec{\Gamma}(u) = \{v\}$ and $|\vec{\Gamma}(v)| \neq 1$ then $d(R_v, R_u) \leq 2T$.
- (i) If $|\vec{\Gamma}(u)| \geq 2$ and $v \in \vec{\Gamma}(u)$, then $d(R_v, R_u) \geq T$.
- (j) u is a leaf if, and only if, $\text{sh}(R_u) \leq Z$.

Before continuing, we make a simple observation about the height, $h(\mathcal{H})$ of a hierarchy \mathcal{H} , by which we mean the maximum distance in $G_{\mathcal{H}}$ of a leaf from the root.

Lemma 5. *Let R be a rectangle, let $Z > 1 > T > 0$, and let \mathcal{H} be a hierarchy for R which is good for (T, Z) . Then*

$$h(\mathcal{H}) \leq \frac{8}{T} \log \left(\frac{\phi(R)}{Z} \right) + 1.$$

Proof. Consider a path P of length $h(\mathcal{H})$ from the root to a leaf u . Let w be the neighbour of u in $G_{\mathcal{H}}$, and note that $\text{sh}(R_w) > Z$. Note also that in every two steps backwards along P , at least one of the dimensions of the corresponding rectangle increases by a factor of at least $1 + T$. Hence one of the dimensions goes up by this factor at least $(h(\mathcal{H}) - 1)/4$ times (on the path from w to the root), and so

$$Z(1 + T)^{(h(\mathcal{H})-1)/4} \leq \phi(R).$$

The result follows by rearranging and using the inequality $\log(1 + T) \geq T/2$, which is valid for all $T \in (0, 1)$. \square

The following key lemma about hierarchies was proved in [25]. Although our definition of hierarchy is slightly different, the proof in our case is identical.

Lemma 6 (Proposition 32 of [25]). *Let $Z > 1 > T > 0$, let R be a rectangle, and suppose A internally spans R . Then there exists a hierarchy \mathcal{H} for R , which is good for (T, Z) , and which is satisfied by A .*

We can now easily deduce, as in Section 10 of [25], our basic bound on the probability of percolation. Given a rectangle R and a pair $(T, Z) \in \mathbb{R}^2$, we write $\mathcal{H}(R, T, Z)$ for the collection of hierarchies for R which are good for (T, Z) .

Recall that $\mathbb{P}_p(I(R))$ and $\mathbb{P}_p(D(S, R))$ denote the probabilities, given $A \sim \text{Bin}(R, p)$, of the events “ R is internally spanned by A ” and “ R is internally spanned by $A \cup S$ ” respectively.

Lemma 7. *Let R be a rectangle in $[n]^2$, let $Z > 1 > T > 0$, let $p > 0$ and let $A \sim \text{Bin}(R, p)$. Then*

$$\mathbb{P}([A] = R) \leq \sum_{\mathcal{H} \in \mathcal{H}(R, T, Z)} \left(\prod_{\vec{\Gamma}(u) = \{v\}} \mathbb{P}_p(D(R_v, R_u)) \right) \prod_{\text{seeds } u} \mathbb{P}_p(I(R_u)).$$

(Above and in subsequent usage, the first product is over all pairs of vertices (u, v) of \mathcal{H} that satisfy the given condition $\vec{\Gamma}(u) = \{v\}$, and the second product is over all seeds u of \mathcal{H} .)

Proof of Lemma 7. By Lemma 6, if A internally spans R then there exists a hierarchy in $\mathcal{H}(R, T, Z)$ which is satisfied by A . Hence the probability that A internally spans R is bounded above by the expected number of such hierarchies. Since the events “ R_u is internally spanned by A ” and $D(R_v, R_u)$ (see (e) and (f) above) are all monotone, and all occur disjointly, the result follows by the van den Berg-Kesten Lemma. \square

We recall the following lemma of Aizenman and Lebowitz [2], which is a standard tool for proving lower bounds on $p_c([n]^d, 2)$.

Lemma 8. *Suppose A internally spans $[n]^2$. Then, for all $1 \leq L \leq n$, there exists a rectangle R , internally spanned by A , with*

$$L \leq \text{long}(R) \leq 2L.$$

We recall also the following bound on $\mathbb{P}_p(D(R, R'))$ from [23].

Lemma 9 (Lemma 5 of [23]). *Let $R \subset R'$ be rectangles of dimensions (a, b) and $(a+s, b+t)$ respectively. Then*

$$\mathbb{P}_p(D(R, R')) \leq \exp\left(-sg(bq) - tg(aq) + 2(g(bq) + g(aq)) + stqe^{2g(bq)+2g(aq)}\right).$$

The following observation is also from [23].

Observation 10 (Lemma 10 of [23]). *Let $a \leq B/q$. Then $e^{2g(aq)} \leq \frac{4B}{aq}$.*

We shall need a couple more definitions in order to rewrite Lemmas 7 and 9 in a more useful form. Let

$$W_g(\mathbf{a}, \mathbf{b}) := \inf_{\gamma: \mathbf{a} \rightarrow \mathbf{b}} \int_{\gamma} (g(y) dx + g(x) dy),$$

where the infimum is taken over all piecewise linear, increasing paths from \mathbf{a} to \mathbf{b} in \mathbb{R}^2 (see Section 6 of [25]). Now, for any two rectangles $R \subset R'$, define

$$U(R, R') = W_g(q \dim(R), q \dim(R')).$$

The following observation is immediate from the definition.

Observation 11 (Proposition 13 of [25]). *Let $R \subset R'$ be rectangles of dimensions (a, b) and $(a+s, b+t)$ respectively. Then*

$$sg(bq) + tg(aq) \geq \frac{1}{q} U(R, R').$$

Let $N(\mathcal{H})$ denote the number of vertices in a hierarchy \mathcal{H} , and $M(\mathcal{H})$ for the number of vertices of \mathcal{H} which have outdegree two. The following technical lemma was proved in [25].

Lemma 12 (Lemma 37 of [25]). *Let \mathcal{H} be a hierarchy for the rectangle R . Then there exists a rectangle $S \subset R$, called the ‘pod’ of \mathcal{H} , such that*

$$\dim(S) \leq \sum_{\text{seeds } u} \dim(R_u)$$

and

$$\sum_{\vec{\Gamma}(v)=\{w\}} U(R_w, R_v) \geq U(S, R) - M(\mathcal{H})qq(Zq).$$

We shall use the following observation to bound $U(S, R)$ from below, and again later in the proof of Theorem 1.

Observation 13. *There exists $C > 0$ such that, for every $0 < a < \infty$, we have*

$$\int_0^a g(z) dz \leq \frac{a}{2} \log \frac{1}{a} + Ca.$$

Proof. Let $\varepsilon > 0$ be such that Observation 4 holds when $z < \varepsilon$. Then, if $a < \varepsilon$ we have

$$\int_0^a g(z) dz \leq \frac{1}{2} \int_0^a -\log z + 2z dz \leq \frac{a}{2} \log \frac{1}{a} + a + a^2,$$

as required. Moreover, since g is decreasing, we have

$$\int_\varepsilon^a g(z) dz \leq ag(\varepsilon),$$

and so the observation follows. \square

Finally, we shall use the following lemma, which follows from Lemma 16 of [25] (see also Lemma 7 of [23]). Recall that we use the notation $O(\cdot)$ to denote the existence of an absolute constant, independent of all variables, such that if we multiply the function in the brackets by this constant then the bound holds.

Lemma 14. *Let $q > 0$ and $S \subset R$, with $\dim(S) = (a, b)$ and $\dim(R) = (A, B)$, where $A \leq B$. If $b \leq A$, then*

$$\frac{1}{q}U(S, R) \geq \frac{2}{q} \int_0^{Aq} g(z) dz + (B - A)g(Aq) - \frac{\phi(S)}{2} \log \frac{2}{\phi(S)q} - O(\phi(S)).$$

If $b > A$, then

$$\frac{1}{q}U(S, R) \geq (A - a)g(bq) + (B - b)g(Aq).$$

Proof. By Lemma 16 of [25], the path integral is minimized by paths which follow the main diagonal as closely as possible. Assuming for simplicity that $a \leq b$, by following the piecewise linear path $(a, b) \rightarrow (b, b) \rightarrow (A, A) \rightarrow (A, B)$ we obtain

$$\frac{1}{q}U(S, R) \geq (b - a)g(bq) + \frac{2}{q} \int_{bq}^{Aq} g(z) dz + (B - A)g(Aq).$$

Now, by Observation 13, we have

$$\frac{2}{q} \int_0^{bq} g(z) dz \leq b \log \frac{1}{bq} + O(b),$$

and by Observation 4, $g(bq) \geq \frac{1}{2} \log(1/bq) - O(1)$. Hence

$$(b - a)g(bq) - \frac{2}{q} \int_0^{bq} g(z) dz \geq -\frac{a + b}{2} \log \frac{1}{bq} - O(b),$$

as required. The inequality for $b > A$ can be obtained by following the path $(a, b) \rightarrow (A, b) \rightarrow (A, B)$, and applying Lemma 16 of [25]. \square

4. THE PROOF OF THEOREM 1

In this section we shall put together the pieces and prove Theorem 1. Recall that, given $p > 0$, we define $q = -\log(1-p) \sim p$.

Proposition 15. *Let $C > 0$ and $\varepsilon > 0$ be constants, let $p = p(C, \varepsilon) > 0$ be sufficiently small, and let R be a rectangle with dimensions (a, b) , where*

$$\frac{\varepsilon}{q} \leq a \leq b \leq \frac{C}{q} \log\left(\frac{1}{q}\right).$$

Let $A \sim \text{Bin}(R, p)$. Then

$$\mathbb{P}_p([A] = R) \leq \exp\left(-\left[\frac{2}{q} \int_0^{aq} g(z) dz + (b-a)g(aq)\right] + \frac{O_C(1)}{\sqrt{q}} \left(\log \frac{1}{q}\right)^3\right).$$

We remark that the constant implicit in the $O_C(1)$ term depends on the constant C , but not on the variables p , a and b (and also not on the constant ε).

Proof. We begin by defining some of the parameters we shall use. First, set $B = C \log(1/q)$, so that $a \leq b \leq B/q$, set $T = \sqrt{q}$, and set

$$Z = \frac{1}{\sqrt{q}} \left(\log \frac{1}{q}\right)^3.$$

Claim: Let $S = S(\mathcal{H})$ denote the pod of a hierarchy \mathcal{H} , given by Lemma 12. Then

$$\mathbb{P}_p([A] = R) \leq \sum_{\mathcal{H} \in \mathcal{H}(R, T, Z)} \exp\left[-\frac{1}{q} U(S, R) + O_C\left(N(\mathcal{H}) \left(\log \frac{1}{q}\right)^2\right)\right] \prod_{\text{seeds } u} \mathbb{P}_p(I(R_u)).$$

In order to prove the claim, first note that by Observation 11 and Lemma 12, the pod $S \subset R$ of \mathcal{H} satisfies

$$\sum_{\vec{\Gamma}(u_i) = \{v_i\}} s_i g(b_i q) + t_i g(a_i q) \geq \frac{1}{q} \sum_{\vec{\Gamma}(u) = \{v\}} U(R_v, R_u) \geq \frac{1}{q} U(S, R) - M(\mathcal{H}) g(Zq),$$

where (a_i, b_i) and $(a_i + s_i, b_i + t_i)$ are the dimensions of R_{v_i} and R_{u_i} respectively.

Now, by the definition of a hierarchy, we have $s_i \leq 2Ta_i$ and $t_i \leq 2Tb_i$ for every pair (u_i, v_i) with $\vec{\Gamma}(u_i) = \{v_i\}$. Recall that $g(z)$ is decreasing, so

$$\max\{g(Zq), g(a_i q), g(b_i q)\} \leq g(q) \leq \log \frac{1}{q},$$

by Observation 4, and that $a_i, b_i \leq B/q$. By Observation 10, it follows that $s_i e^{2g(a_i q)} \leq 4Bs_i/a_i q \leq 8BT/q$, and similarly for b_i . Thus

$$g(Zq) + 2g(b_i q) + 2g(a_i q) + s_i t_i q e^{2g(b_i q) + 2g(a_i q)} \leq 5 \log \frac{1}{q} + O\left(\frac{B^2 T^2}{q}\right),$$

and hence, since $T^2 = q$, $B = O_C(\log(1/q))$ and $M(\mathcal{H}) \leq N(\mathcal{H})$,

$$\mathcal{M}(\mathcal{H})g(Zq) + \sum_{\bar{\Gamma}(u)=\{v\}} \left(2g(bq) + 2g(aq) + stqe^{2g(bq)+2g(aq)} \right) = O_C \left(N(\mathcal{H}) \left(\log \frac{1}{q} \right)^2 \right).$$

Hence, by Lemma 9, we have

$$\prod_{\bar{\Gamma}(u)=\{v\}} \mathbb{P}_p(D(R_v, R_u)) \leq \exp \left[-\frac{1}{q}U(S, R) + O_C \left(N(\mathcal{H}) \left(\log \frac{1}{q} \right)^2 \right) \right],$$

and so the claim follows by Lemma 7.

Our problem now is that there are too many hierarchies: there could be as many as $2^{1/\sqrt{q}}$ vertices in $G_{\mathcal{H}}$, and for each vertex u we have many choices for the rectangle R_u . However, most of these hierarchies have many seeds, and those with many *large* seeds have rather small weight in the sum. This turns out to be the key idea in the proof.

Indeed, let us define a *large seed* to be one with $\phi(R_u) \geq Z/3$, and note that every (non-seed) vertex of \mathcal{H} lies above at least one large seed. Let the number of large seeds in a hierarchy \mathcal{H} be denoted $m = m(\mathcal{H})$. Observe that, by Lemma 5, \mathcal{H} has height at most $(10/\sqrt{q}) \log(1/q)$, and hence that the number of vertices $N(\mathcal{H})$ in $G_{\mathcal{H}}$ satisfies

$$N(\mathcal{H}) \leq 2m \cdot h(\mathcal{H}) = O \left(\frac{m}{\sqrt{q}} \log \frac{1}{q} \right). \quad (2)$$

Therefore, the number of hierarchies with m large seeds is at most

$$\left(\frac{B}{q} \right)^{4N(\mathcal{H})} \leq \exp \left(O(1) \frac{m}{\sqrt{q}} \left(\log \frac{1}{q} \right)^2 \right). \quad (3)$$

Now, for each hierarchy \mathcal{H} , define

$$X = X(\mathcal{H}) := \sum_{\text{seeds } u} \phi(R_u),$$

and note that $X(\mathcal{H}) \geq \frac{m(\mathcal{H})Z}{3}$, and that $\phi(S(\mathcal{H})) \leq X(\mathcal{H})$, by Lemma 12. By Lemma 2, for every seed R_u we have

$$\mathbb{P}_p(I(R_u)) \leq 3^{\phi(R_u)} \exp \left(-\phi(R_u)g(Zq) \right),$$

since $\text{sh}(R_u) \leq Z = o(1/q)$ as $q \rightarrow 0$, and $g(z)$ is decreasing in z . Thus

$$\prod_{\text{seeds } u} \mathbb{P}_p(I(R_u)) \leq \prod_{\text{seeds } u} 3^{\phi(R_u)} \exp \left(-\phi(R_u)g(Zq) \right) \leq 3^X \exp \left(-Xg(Zq) \right). \quad (4)$$

We split into two cases. The first is easier to handle, and we shall not have to approximate too carefully; in the second the calculation is much tighter.

Case 1: $\lg(S) > a$.

We have $a < \phi(S) \leq X$, by Lemma 12, and $\frac{1}{q}U(S, R) \geq (b - X)g(aq)$, by Lemma 14. Hence, by the claim, (2) and (4),

$$\mathbb{P}([A] = R) \leq \sum_{\mathcal{H} \in \mathcal{H}(R, T, Z)} 3^X \exp \left(-(b - X)g(aq) + O_C \left(\frac{m}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3 \right) - Xg(Zq) \right).$$

Now, by Observation 4, and since $X \geq mZ/3$ and $a/Z \geq q^{-1/3}$, we have

$$X(g(Zq) - g(aq)) \geq \frac{X}{7} \log \left(\frac{1}{q} \right) = \frac{1}{o(1)} mZ = \frac{1}{o(1)} \frac{m}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3$$

as $q \rightarrow 0$. Using (3), it follows that

$$\mathbb{P}([A] = R) \leq \sum_{\mathcal{H}} \exp \left(-bg(aq) - \frac{X}{8} \log \frac{1}{q} \right) \leq \frac{1}{q} \exp \left(-\frac{2}{q} \int_0^{aq} g(z) dz - bg(aq) \right),$$

as required. The final inequality follows from Observation 13, since

$$\frac{2}{q} \int_0^{aq} g(z) dz \leq a \log \left(\frac{1}{aq} \right) + O(a) = o \left(X \log \frac{1}{q} \right)$$

as $q \rightarrow 0$ (recall that $aq \geq \varepsilon$ and $a < X$). Note also that $\sum_{m > 1/q} e^{-mZ} \leq e^{-q^{-3/2}}$.

Case 2: $\lg(S) \leq a$.

By Lemma 14, and since $\phi(S) \leq X$, we have

$$\frac{1}{q}U(S, R) \geq \frac{2}{q} \int_0^{aq} g(z) dz + (b - a)g(aq) - \frac{X}{2} \log \frac{1}{Xq} - O(X).$$

Hence, by the claim, (2) and (4), we have

$$\begin{aligned} \mathbb{P}_p([A] = R) \leq \sum_{\mathcal{H} \in \mathcal{H}(R, T, Z)} 3^X \exp \left[-\frac{2}{q} \int_0^{aq} g(z) dz - (b - a)g(aq) + \frac{X}{2} \log \frac{1}{Xq} \right. \\ \left. + O(X) + O_C \left(\frac{m}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3 \right) - Xg(Zq) \right]. \end{aligned}$$

By Observation 4, this is at most

$$\sum_{\mathcal{H}} \exp \left[-\frac{2}{q} \int_0^{aq} g(z) dz - (b - a)g(aq) - \frac{X}{2} \log \frac{X}{C'Z} + O_C \left(\frac{m}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3 \right) \right],$$

for some constant $C' > 0$. Now, note that $\frac{X}{2} \log \frac{X}{C'Z}$ is increasing in X , and recall that $X \geq \frac{mZ}{3}$ and $Z = \frac{1}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3$. Thus

$$\begin{aligned} -\frac{X}{2} \log \frac{X}{C'Z} + O_C \left(\frac{m}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3 \right) &\leq -\frac{mZ}{6} \log \frac{m}{3C'} + O_C \left(\frac{m}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3 \right) \\ &\leq \frac{O_C(1)}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3 - \frac{m}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3. \end{aligned}$$

Hence, using (3), and summing over m , we obtain

$$\mathbb{P}_p([A] = R) \leq \exp \left[-\frac{2}{q} \int_0^{aq} g(z) dz - (b-a)g(aq) + \frac{O_C(1)}{\sqrt{q}} \left(\log \frac{1}{q} \right)^3 \right],$$

as required. \square

Before deducing Theorem 1 from Proposition 15, we need to recall the following fact from [25], and to make an easy observation.

Lemma 16 (Proposition 5 of [25]).

$$\int_0^\infty g(z) dz = \frac{\pi^2}{18}.$$

The following observation follows almost immediately from Lemma 16.

Observation 17. *Let $p > 0$ be sufficiently small, and let $a, b \in \mathbb{R}$, with $a \leq b$ and $b \geq B/2p$, where $B = 10 \log(1/p)$. Then*

$$\frac{2}{q} \int_0^{aq} g(z) dz + (b-a)g(aq) \geq \frac{2\lambda}{q} - 1,$$

where $\lambda = \pi^2/18$.

Proof. If $a \leq B/4p$, then this follows since $\int_{aq}^\infty g(z) dz = O(g(aq))$, uniformly over $a \in (0, \infty)$, and so

$$(b-a)g(aq) - \frac{2}{q} \int_{aq}^\infty g(z) dz \geq \left(\frac{B}{4p} \right) g(aq) - O \left(\frac{g(aq)}{q} \right) > 0.$$

If $a \geq B/4p$ then it holds because $g(z) \leq 2e^{-z}$ for z large, and so

$$\frac{2}{q} \int_{aq}^\infty g(z) dz \leq \frac{4}{q} e^{-aq} \leq \frac{4}{q} e^{-B/5} \leq 1,$$

as required. \square

Finally, we deduce Theorem 1 from Proposition 15.

Proof of Theorem 1. Let $C > 0$ be a large constant to be chosen later, let $n \in \mathbb{N}$, and let

$$p < \frac{\pi^2}{18 \log n} - \frac{C(\log \log n)^3}{(\log n)^{3/2}}.$$

Note that $q = -\log(1-p) < p + p^2$, and so q also satisfies this inequality (possibly with a slightly different constant C).

Let $A \sim \text{Bin}([n]^2, p)$, and suppose that A percolates. Then, by Lemma 8, there exists a rectangle $R \subset [n]^2$, which is internally spanned by A , and with $B/2p \leq \lg(R) \leq B/p$, where $B = 10 \log(1/p)$. Let $\dim(R) = (a, b)$, and assume without loss of generality that $a \leq b$. There are at most $n^2(B/p)^2$ potential such rectangles, and each is internally spanned with probability at most

$$\mathbb{P}_p([A \cap R] = R) \leq \exp\left(-\left[\frac{2}{q} \int_0^{aq} g(z) dz + (b-a)g(aq)\right] + \frac{O(1)}{\sqrt{q}} \left(\log \frac{1}{q}\right)^3\right),$$

if $\text{sh}(R) \geq 1/q$, by Proposition 15, and with probability at most

$$e^{-bg(aq)} \leq e^{-B/10p} = p^{1/p} \leq \left(\frac{1}{n}\right)^{100}$$

if C is sufficiently large, and $\text{sh}(R) \leq 1/q$.

By Observation 17, and using the identity $\frac{1}{a-b} = \frac{1}{a} + \frac{b}{a(a-b)}$, this gives, as $n \rightarrow \infty$,

$$\begin{aligned} \mathbb{P}([A] = [n]^2) &\leq n^2(B/p)^2 \exp\left(-\frac{2\lambda}{q} + \frac{O(1)}{\sqrt{q}} \left(\log \frac{1}{q}\right)^3\right) \\ &\leq n^2(B/p)^2 \exp\left(-2 \log n - \frac{C}{\lambda} (\log \log n)^3 \sqrt{\log n} + \frac{O(1)}{\sqrt{q}} \left(\log \frac{1}{q}\right)^3\right) \\ &\leq n^2(\log n)^3 \exp\left(-2 \log n - (\log \log n)^3 \sqrt{\log n}\right) \rightarrow 0 \end{aligned}$$

if C is sufficiently large, as required. \square

5. EXTENSIONS AND OPEN QUESTIONS

In this paper we have studied bootstrap percolation on one particular graph, the two-dimensional grid with nearest-neighbour bonds. It is natural to ask whether our method can be applied to bootstrap percolation on other graphs; here we shall discuss two such possible generalizations.

The most obvious (and most extensively studied) generalization is to consider bootstrap percolation in d dimensions (i.e., on the graph $[n]^d$), with nearest neighbour interaction and threshold $2 \leq r \leq d$ (as studied in, for example, [2, 5, 6, 7, 16, 32]). The sharp metastability thresholds for these models (with d fixed, and as $n \rightarrow \infty$) will be determined in [8], and it is likely that the methods of this paper (and those of [22]) could be adapted to give improved bounds in the case of $r = 2$ and general d .

Problem 1. Give better bounds on the second term in the asymptotic expansion of $p_c([n]^d, 2)$ as $n \rightarrow \infty$.

The case $r \geq 3$ is more complicated, and the following problem is likely to be difficult.

Problem 2. Give good bounds on the second term in the asymptotic expansion of $p_c([n]^3, 3)$ as $n \rightarrow \infty$.

A second natural generalization is to consider bootstrap percolation in two dimensions, but with a different update rule. For example, in the ‘modified’ bootstrap process (see [26]), a vertex is infected if at least one of its neighbours in *each* dimension is already infected; in the ‘ k -cross’ process (see [27, 14]), a vertex v is infected if at least k vertices in the cross-shaped set

$$\bigcup_{0 \neq j \in [-k+1, k-1]} \{v + (0, j), v + (j, 0)\}$$

are previously infected; and in the Froböse process (introduced by Froböse [21] in 1989) a site of $[n]^2$ is infected if it has one already-infected neighbour in each dimension, along with the next-nearest neighbour in the corner between them. In general (see [18]), one could consider an arbitrary neighbourhood $N(v)$ of each vertex v , an arbitrary (monotone) family $\mathcal{A}(v)$ of subsets of $N(v)$, and say that v becomes infected if the already-infected subset of its neighbours is in $\mathcal{A}(v)$.

Holroyd [25] (see also [26]) determined the sharp threshold for the modified and Froböse models, and Holroyd, Liggett and Romik [27] did so for the k -cross process for all fixed $k \in \mathbb{N}$. Moreover, Duminil-Copin and Holroyd [18] have recently shown, for a large family of such models (including all of the examples above, and other similar models), that there exists a sharp metastability threshold. It is not unreasonable to hope that our method (together with that of [22]) might yield improved bounds on the critical probability for a more general collection of bootstrap processes, of the type considered in [18]. Indeed, for two of the processes described above this is the case.

Let $p_c^{(F)}([n]^2)$ denote the critical probability for percolation in the Froböse process on $[n]^2$, and let $p_c^{(+)}([n]^2, k)$ denote the critical probability for percolation in the k -cross process. The upper bounds in the following theorem were proved by Gravner and Holroyd [22] (for the Froböse model) and by Bringmann and Mahlburg [14] (for the k -cross process). The lower bounds follow by the methods of this paper.

Theorem 18.

$$p_c^{(F)}([n]^2) = \frac{\pi^2}{6 \log n} - \frac{1}{(\log n)^{3/2+o(1)}}.$$

as $n \rightarrow \infty$. Let $k \in \mathbb{N}$, and let $\lambda_k = \pi^2/3k(k+1)$. Then

$$p_c^{(+)}([n]^2, k) = \frac{\lambda_k}{\log n} - \frac{1}{(\log n)^{3/2+o(1)}}.$$

In fact the bounds we prove (and those from [22, 14]) are a little stronger than those stated above; they are like the bounds in Theorem 1.

Sketch of proof of Theorem 18. For the first part, it suffices to show that (in the Froböse process) on $R = [m] \times [n]$, all spanning sets have size at least $m + n - 1$. The theorem then follows in exactly the same way as Theorem 1.

We shall give two proofs that if $[A] = R$ then $|A| \geq m + n - 1$. The first is standard, using Proposition 30 of [25] (see also [3], or Lemma 12 of [4]) and induction on $\phi(R)$. For the second, consider the (bipartite) graph G whose vertices are the rows and columns of R , with an edge from row x to column y if and only if $(x, y) \in A$.

To prove that G has at least $m + n - 1$ edges, we shall show that it is connected. Indeed, if G is not connected then exists a set of rows X and a set of columns Y such that $A \subset S = (X \cap Y) \cup (X^c \cap Y^c)$. But then $[S] = S \neq R$, so A does not percolate, as required.

For the second part, we need the following idea from [27]: first couple the k -cross process with an ‘enhanced process’ (see [27], Section 5) in which the closed sets are rectangles. In the enhanced process the minimum number of sites required to infect an $[m] \times [n]$ rectangle is about $(m + n)/k$, which is also the typical number required. (To prove this, apply the standard proof, by induction on $m + n$.) The result now follows by the proof of Theorem 1. \square

Gravner and Holroyd [22] also improved the upper bounds for the modified process. However, the proof of Theorem 1 does not work for the modified process, since we do not have a result analogous to Lemma 2. In particular, it is possible to internally span an $m \times n$ rectangle with $\max\{m, n\}$ infected sites, but the proportion of such minimal-size sets which percolate is very small.

Let $p_c^{(M)}([n]^d)$ denote the critical probability for percolation in the modified bootstrap process on the graph $[n]^d$, i.e., the infimum over p such that the probability of percolation is at least $1/2$. We have the following conjecture; it is the analogue of Conjecture 1 for the modified process.

Conjecture 2. *As $n \rightarrow \infty$,*

$$p_c^{(M)}([n]^2) = \frac{\pi^2}{6 \log n} - \frac{1}{(\log n)^{3/2+o(1)}}.$$

Given a rectangle R , we say that a set $A \subset R$ is a *minimal percolating set* if A spans R , but no proper subset of A does so (see [30], for example). Given $m \geq n$ and $x \geq 0$, let $F(m, n, x)$ denote the number of minimal percolating sets of size $m + x$ in modified bootstrap percolation on $R = [m] \times [n]$. We remark that Conjecture 2 would follow from the method of this paper, together with following bound:

$$F(m, n, x) \leq n^{m-n+2x+o(n)}.$$

We remark that even if we restrict ourselves to ‘threshold’ models, in which a vertex is infected if at least r elements of its neighbourhood are infected, we still run into similar problems. Indeed, consider the model in which a vertex is infected if at least four of its eight neighbours (including diagonals) are infected. A typical seed R is shaped like an octagon, and the number of infected sites used to fill R (in the random process) is roughly

$\phi(R)$ (which we define to be the number of external vertices plus the number of external edges), while the minimal number required to span R is only $\phi(R)/2$.

Finally, returning to the standard bootstrap process, recall that Theorem 1 determines the second term of $p_c([n]^2, 2)$ up to a poly($\log \log n$)-factor. We conjecture that this error term can be removed.

Conjecture 3. *As $n \rightarrow \infty$,*

$$p_c([n]^2, 2) = \frac{\pi^2}{18 \log n} - \Theta\left(\frac{1}{(\log n)^{3/2}}\right).$$

REFERENCES

- [1] J. Adler and U. Lev, Bootstrap Percolation: visualizations and applications, *Braz. J. Phys.*, **33** (2003), 641–644.
- [2] M. Aizenman and J.L. Lebowitz, Metastability effects in bootstrap percolation, *J. Phys. A.*, **21** (1988) 3801–3813.
- [3] J. Balogh, Graph Parameters and Bootstrap Percolation, Ph.D. Dissertation, Memphis, 2001.
- [4] J. Balogh and B. Bollobás, Bootstrap percolation on the hypercube, *Prob. Theory Rel. Fields*, **134** (2006), 624–648.
- [5] J. Balogh, B. Bollobás and R. Morris, Majority bootstrap percolation on the hypercube, *Combinatorics, Probability and Computing*, **18** (2009), issue 1-2, 17–51.
- [6] J. Balogh, B. Bollobás and R. Morris, Bootstrap percolation in three dimensions, *Annals of Probability*, **37** (4) (2009), 1329–1380.
- [7] J. Balogh, B. Bollobás and R. Morris, Bootstrap percolation in high dimensions, submitted.
- [8] J. Balogh, B. Bollobás, H. Duminil-Copin and R. Morris, The sharp metastability threshold for r -neighbour bootstrap percolation. In preparation.
- [9] J. Balogh, Y. Peres and G. Pete, Bootstrap percolation on infinite trees and non-amenable groups, *Combinatorics, Probability and Computing*, **15** (2006), 715–730.
- [10] J. Balogh and B. Pittel, Bootstrap percolation on random regular graphs. *Random Structures and Algorithms*, **30** (2007), 257–286.
- [11] J. van den Berg and H. Kesten, Inequalities with applications to percolation and reliability, *J. Appl. Probab.*, **22** (1985), 556–589.
- [12] M. Biskup and R.H. Schonmann, Metastable behavior for bootstrap percolation on regular trees, *J. Statist. Phys.* **136** (2009), no. 4, 667–676.
- [13] B. Bollobás, *The Art of Mathematics: Coffee Time in Memphis*, CUP, Cambridge 2006.
- [14] K. Bringmann and K. Mahlborg, Improved bounds on metastability thresholds and probabilities for generalized bootstrap percolation, arXiv:1001.1977
- [15] R. Cerf and E. N. M. Cirillo, Finite size scaling in three-dimensional bootstrap percolation, *Ann. Probab.*, **27** (1999), 1837–1850.
- [16] R. Cerf and F. Manzo, The threshold regime of finite volume bootstrap percolation, *Stochastic Proc. Appl.*, **101** (2002), 69–82.
- [17] J. Chalupa, P. L. Leath and G. R. Reich, Bootstrap percolation on a Bethe lattice, *J. Phys. C.*, **12** (1979), L31–L35.
- [18] H. Duminil-Copin and A. Holroyd, Sharp metastability for threshold growth models. In preparation.
- [19] L.R. Fontes, R.H. Schonmann, Bootstrap percolation on homogeneous trees has 2 phase transitions, *J. Stat. Phys.*, **132** (2008), 839–861.
- [20] L.R. Fontes, R.H. Schonmann and V. Sidoravicius, Stretched Exponential Fixation in Stochastic Ising Models at Zero Temperature, *Commun. Math. Phys.*, **228** (2002), 495–518.

- [21] K. Froböse. Finite-size effects in a cellular automaton for diffusion, *J. Statist. Phys.*, **55** (1989) (5-6), 1285–1292.
- [22] J. Gravner and A.E. Holroyd, Slow convergence in bootstrap percolation, *Ann. Appl. Prob.*, **18** (2008), 909–928.
- [23] J. Gravner and A.E. Holroyd, Local bootstrap percolation, *Electron. J. Probability*, **14** (2009), Paper 14, 385–399.
- [24] P. De Gregorio, A. Lawlor, P. Bradley and K.A. Dawson, Exact solution of a jamming transition: closed equations for a bootstrap percolation problem. *Proc. Natl. Acad. Sci. USA*, **102** (2005), no. 16, 5669–5673 (electronic).
- [25] A. Holroyd, Sharp Metastability Threshold for Two-Dimensional Bootstrap Percolation, *Prob. Th. Rel. Fields*, **125** (2003), 195–224.
- [26] A. Holroyd, The Metastability Threshold for Modified Bootstrap Percolation in d Dimensions, *Electron. J. Probability*, **11** (2006), Paper 17, 418–433.
- [27] A.E. Holroyd, T.M. Liggett and D. Romik. Integrals, partitions, and cellular automata, *Trans. Amer. Math. Soc.*, **356** (2004) (8), 3349–3368.
- [28] S. Janson, On percolation in Random Graphs with given vertex degrees. arXiv:0804.1656.
- [29] S. Janson, T. Łuczak, T. Turova and T. Vallier, personal communication.
- [30] R. Morris, Minimal percolating sets in bootstrap percolation, *Electron. J. Combin.*, **16** (2009), Research Paper 2, 20pp.
- [31] R. Morris, Zero-temperature Glauber dynamics on \mathbb{Z}^d , to appear in *Prob. Theory Rel. Fields*.
- [32] R.H. Schonmann, On the behaviour of some cellular automata related to bootstrap percolation, *Ann. Prob.*, **20** (1992), 174–193.
- [33] P. Winkler, *Mathematical puzzles: a connoisseurs collection*, A K Peters Ltd., Natick, MA, 2004.

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