# A NON-MEASURABLE SET FROM COIN-FLIPS

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To motivate the elaborate machinery of measure theory, it is desirable to have an example of a set which is not measurable in some natural space. The usual example is the *Vitali set*, obtained by picking one representative from each equivalence class of $\mathbb{R}$ induced by the relation $x \sim y$ iff $x - y \in \mathbb{Q}$. The translation-invariance of Lebesgue measure implies that the resulting set is not Lebesgue-measurable [4]. By the Solovay Theorem [3], one cannot construct such a set in Zermelo-Frankel set theory without appealing to the axiom of choice. In this note we give a variant construction in the language of probability theory, using the axiom of choice in the guise of the well-ordering principle [5]. For other constructions see [2, Ch. 5].

Consider the measure space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{0,1\}^{\mathbb{Z}}$, and $\mathcal{F}$ is the product $\sigma$-algebra, and $\mathbb{P}$ is the product measure $(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\mathbb{Z}}$. This is the probability space for a sequence of independent fair coin flips indexed by $\mathbb{Z}$. It is well-known that $(\Omega, \mathcal{F}, \mathbb{P})$ is isomorphic up to null sets to Lebesgue measure on $[0,1)$, via binary expansion [1, Theorem 3.19].

**Theorem 1.** *There exists a set $S \subset \Omega$ which is not $\mathcal{F}$-measurable.*

The **shift** $T$ acts on integers via $Tx := x+1$, on configurations $\omega \in \Omega$ via $(T\omega)(x) := \omega(x-1)$ and on subsets of $\Omega$ via $T(E) := \{T\omega : \omega \in E\}$. We shall see that the set $S$ is in fact not $\mathcal{F}'$-measurable in any $(\Omega, \mathcal{F}', \mathbb{P}')$ where $\mathcal{F}' \supseteq \mathcal{F}$ and $\mathbb{P}'$ is shift-invariant and non-atomic.

Consider a function $X : \Omega \to \mathbb{Z} \cup \{\Delta\}$. We call $X$ **almost everywhere defined** if $\mathbb{P}(X^{-1}\{\Delta\}) = 0$. We call $X$ **shift-equivariant** if

$$X(T(\omega)) = T(X(\omega)) \quad \text{for all } \omega \in \Omega$$

(where $T(\Delta) := \Delta$). Theorem 1 is an immediate consequence of the following two facts.

**Lemma 2.** *There does not exist an $\mathcal{F}$-measurable, a.e. defined, shift-equivariant function $X : \Omega \to \mathbb{Z} \cup \{\Delta\}$.*

**Lemma 3.** *There exists an a.e. defined, shift-equivariant function $X : \Omega \to \mathbb{Z} \cup \{\Delta\}$.*

Proof of Lemma 2. Suppose $X$ is such a function. We adopt the usual probabilistic convention that $\{X \in A\}$ is shorthand for $\{\omega \in \Omega : X(\omega) \in A\}$. Since $X$ is shift-equivariant and $\mathbb{P}$ is shift-invariant (by uniqueness of extension [1, Lemma 1.17]) we have for each $x \in \mathbb{Z}$,

$$\mathbb{P}(X = x) = \mathbb{P}\big(T^{-x}\{X = 0\}\big) = \mathbb{P}(X = 0).$$

Hence

$$\mathbb{P}(X \neq \Delta) = \mathbb{P}\Big(\bigcup_{x \in \mathbb{Z}} \{X = x\}\Big) = \sum_{x \in \mathbb{Z}} \mathbb{P}(X = 0) = 0 \text{ or } \infty,$$

which contradicts $\mathbb{P}(X \neq \Delta) = 1$. □

Proof of Lemma 3. Say $\omega$ is **periodic** if $T^x\omega = \omega$ for some $x \in \mathbb{Z}$. If $\omega$ is not periodic then the configurations $(T^x\omega : x \in \mathbb{Z})$ are all distinct. Fix a well-ordering of $\Omega$ and define the function

$$X(\omega) := \begin{cases} \Delta & \text{if } \omega \text{ is periodic;} \\ \operatorname*{argmin}_{x \in \mathbb{Z}} T^{-x}\omega & \text{otherwise.} \end{cases}$$

(Recall that the argmin of a function is the argument at which its minimum is attained). We can think of $X(\omega)$ as the vantage point from which the configuration appears least. Then $X$ is clearly shift-equivariant, and it is a.e. defined since there are only countably many periodic configurations. □

## References

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