Knotted Paths in Percolation

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Abstract

We study the topology of doubly-infinite paths in the bond percolation model on the three-dimensional cubic lattice. We propose a natural definition of a knotted doubly-infinite path. We prove the existence of a critical probability p_k satisfying $p_c < p_k < 1$ (where p_c is the usual percolation critical probability), such that for $p_c < p < p_k$, all doubly-infinite open paths are knotted, while for $p > p_k$ there are unknotted doubly-infinite paths.

1 Introduction

Knotting of random paths has important applications in polymer science, and has been extensively studied. Previous work has involved knotting probabilities of finite self-avoiding walks and polygons, chosen according to various random mechanisms. For details, and for information on the physical applications, the reader is referred to the articles in [10], for example. For more information on knot theory see [9].

Here we consider a closely related problem. In the percolation model, edges of the infinite three-dimensional cubic lattice are declared *open* with probability p, or *closed* with probability $1 - p$, independently for different edges. (For more details of percolation see [2]). We consider the question: when do there exist open knotted or unknotted doubly-infinite paths?

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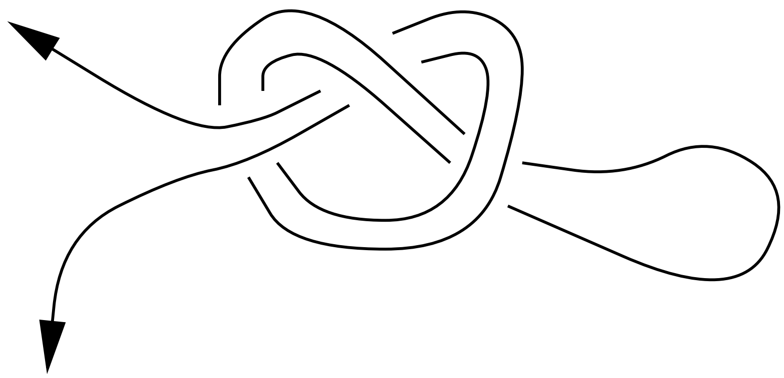

Figure 1: This path is unknotted.

As in the case of entanglement (see [3],[5],[6],[8]) it is not immediately obvious how to give a 'correct' definition of a knotted doubly-infinite path. The situation is complicated by the possibility that a path may 'double back' to untie a potential knot, as in Figure 1. We shall not pursue the question of possible definitions in detail; instead we shall give one natural definition, noting that there may be others.

Standard results imply that when p is greater than the percolation critical probability p_c, there exist open doubly-infinite paths. We shall prove that for p sufficiently close to p_c all such paths are knotted, while for p sufficiently close to 1 there are unknotted doubly-infinite paths.

2 Notation and Results

We start with some definitions. The three-dimensional cubic lattice is the graph with vertex set $\mathbb{Z}^3$ and edge set

$$\mathbb{L} = \{\{x, y\} \subseteq \mathbb{Z}^3 : \|x - y\| = 1\}$$

where $\| \cdot - \cdot \|$ denotes Euclidean distance. The *origin* is the vertex $O = (0, 0, 0) \in \mathbb{Z}^3$. In the bond percolation model with parameter p, each edge in $\mathbb{L}$ is declared *open* with probability p, and *closed* otherwise, independently for different edges. More formally, we consider the product probability measure P_p on the probability space $\{0, 1\}^{\mathbb{L}}$. An element ω of the probability space is called a *configuration*, and an edge $e \in \mathbb{L}$ is said to be *open* if $\omega(e) = 1$

and *closed* if $\omega(e) = 0$. We write $W = W(\omega)$ for the random set of all open edges.

A *finite path* is a non-empty set of edges of the form $\{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{r-1}, x_r\}\}$, and a *doubly-infinite path* is a set of edges of the form $\{\ldots, \{x_{-1}, x_0\}, \{x_0, x_1\}, \{x_1, x_2\}, \ldots\}$, where in both cases the x_i are pairwise distinct vertices. A *subpath* is a subset of a path which is itself a path.

Percolation theory is concerned with the existence of infinite connected components. We define

$$\theta(p) = P_p(W \text{ has an infinite connected component containing } O)$$

and

$$p_c = \sup\{p : \theta(p) = 0\}.$$

It is known (by the results in [4], for example) that for all $p > p_c$ there exist doubly-infinite open paths almost surely. For more information on percolation see [2].

Our aim here is to study knotting of paths, and for this we require the following topological definitions. A *ball* B is a subset of $\mathbb{R}^3$ which is homeomorphic to $\{x \in \mathbb{R}^3 : \|x\| \leq 1\}$, and the *boundary* ∂B of a ball is the image of $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ under such a homeomorphism. Similarly an arc α is a subset of $\mathbb{R}^3$ homeomorphic to $[-1, 1] \times \{0\}^2$, and $\partial\alpha$ is the image of $\{-1, 1\} \times \{0\}^2$. The following definitions relating to ball-arc pairs are standard; for more details see [9]. A *ball-arc pair* is a pair (B, α), where B is a ball and α is an arc, such that $\alpha \subseteq B$ and $\alpha \cap \partial B = \partial\alpha$. Two ball-arc pairs (B, α) and (B', α') are *equivalent* if there is a homeomorphism from B to B' which maps α to α'. A ball-arc pair is said to be *unknotted* if it is equivalent to the ball-arc pair $([-1, 1]^3, [-1, 1] \times \{0\}^2)$, and *knotted* otherwise. (Note that *any* arc forms an unknotted ball-arc pair with some ball; see [9] for more details).

For an edge $e = \{x, y\} \in \mathbb{L}$ we denote by $[e]$ the closed line segment

$$[e] = \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\} \subseteq \mathbb{R}^3.$$

For a set of edges G we write $[G] = \cup_{e \in G}[e] \subseteq \mathbb{R}^3$. By a *block* we mean a ball of the form $[a, b] \times [c, d] \times [e, f]$, where $a, \ldots, f$ are integers. Let F be a *finite* path. We say F is *neat* if there exists a block B such that $(B, [F])$ is an unknotted ball-arc pair. We say that a doubly-infinite path G is *unknotted* if every finite subpath of G is a subpath of some neat finite subpath of G, and *knotted* otherwise.

We now define

$$\kappa(p) = P_p(\text{there is an open unknotted doubly-infinite path containing } O).$$

It is easy to see that κ is an increasing function, so we define

$$p_k = \sup\{p : \kappa(p) = 0\}.$$

Theorem. *We have*

$$p_c < p_k < 1.$$

It follows from the theorem that if $p_c < p < p_k$, then every doubly-infinite path is knotted almost surely.

3 Proof of Theorem

We begin with the latter inequality of the theorem. We say that a doubly-infinite path $\{\ldots, \{x_{-1}, x_0\}, \{x_0, x_1\}, \ldots\}$ is *oriented* if $(x_{i+1})_j \geq (x_i)_j$ for all i and each $j = 1, 2, 3$, where $(x_i)_j$ denotes the j-coordinate of the 3-vector x_i. Standard results imply that for p sufficiently close to unity, O is contained in an open oriented doubly-infinite path with positive probability (see [2], Section 12.8). The inequality $p_k < 1$ therefore follows from the observation (which we justify below) that every oriented doubly-infinite path is unknotted.

To justify the claim above, note that it is sufficient to prove that any finite subpath of an oriented doubly-infinite path is neat. Let $F = \{\{x_0, x_1\}, \ldots, \{x_{r-1}, x_r\}\}$ be such a path. Clearly we may find a block B such that $(B, [F])$ is a ball-arc pair (we start with the block having opposite corners x_0 and x_r, and then enlarge it to ensure that $[F] \cap \partial B = \partial[F] = \{x_0, x_r\}$). For $x \in \mathbb{R}^3$ define $\phi(x) = x_1 + x_2 + x_3$. Note that $\phi(x_i)$ is strictly monotonic in i, increasing (or decreasing) by 1 as i increased by 1. Let L be the straight line segment joining x_0 and x_r. It is straightforward to show that $(B, [F])$ is equivalent to (B, L); there is a suitable piecewise-linear homeomorphism which preserves $\phi(x)$ for all $x \in B$, and is the identity on ∂B. It is now easily seen (by applying a further homeomorphism) that (B, L) is an unknotted ball-arc pair, and hence $(B, [F])$ is also.

We now turn to the former inequality of the theorem. Let $C = [0, 4] \times [0, 5] \times [0, 4]$, let H be the set of all edges of $\mathbb{L}$ having both vertices in C,

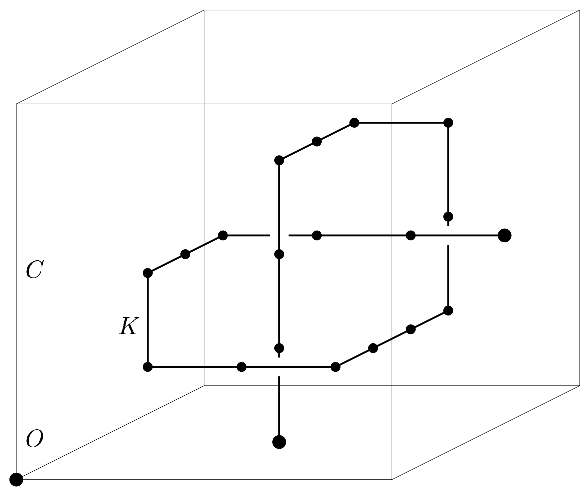

Figure 2: The path K. The ends of the path lie on the boundary of the block C, while all the other vertices lie in its interior.

and let K be the subset of H illustrated in Figure 2 (the outline of C is also illustrated). Standard tools of knot theory may be used to show that $(C, [K])$ is a knotted ball-arc pair (for example, using the Jones polynomial, see [9]). We define a 'diminishment' of W as follows. Given ω, define

$$W' = W \setminus \bigcup_{\substack{x \in \mathbb{Z}^3: \\ W \cap (H+x) = K+x}} (K + x);$$

that is, W' is obtained from W by deleting translated copies of Figure 2 wherever they occur.

The following is a consequence of a slight modification of results in [1]. There exists an interval $[p_1, p_2]$ where $p_1 < p_2$ such that for $p \in [p_1, p_2]$ we have

$$P_p(W \text{ has an infinite connected component}) = 1 \qquad (1)$$

but

$$P_p(W' \text{ has an infinite connected component}) = 0. \qquad (2)$$

The main result in [1] is for 'enhancements' – systematic alterations involving the addition of edges, whereas the construction of W' is a 'diminishment' involving removal of edges. The necessary modifications to the proof in [1] are straightforward. A diminishment was also used in [7]; see also [2], p. 65.

Now, (1) implies that $p_c \le p_1$. And (2) implies that, P_p-a.s. for $p \in [p_1, p_2]$, W' contains no unknotted doubly-infinite path. We shall show that this in turn implies that W has no unknotted doubly-infinite path, and therefore $p_2 \le p_k$, establishing the required inequality.

We must show that there exists no ω for which W contains an unknotted doubly-infinite path but W' does not. Suppose on the contrary that for some ω, U is an unknotted doubly-infinite path which is a subset of W but not of W'. Clearly, U must have a subpath of the form $K + x$, and without loss of generality we may assume that $x = O$, so that $K \subseteq U$ and $W \cap H = K$. Since U is unknotted, K must lie in a neat subpath of U, so consider a block B and a finite path L satisfying $K \subseteq L \subseteq U$ such that $(B, [L])$ is an unknotted ball-arc pair. We shall use standard tools from knot theory to show that this is impossible; detailed justification of some of the steps may be found in [9]. First add a 'point at infinity' to $\mathbb{R}^3$ making it into a 3-sphere. For any ball A, we write $\widehat{A}$ for the closure of its complement in $\mathbb{R}^3 \cup \{\infty\}$; this is also a ball. Now, since $K \subseteq L$, it may be seen by inspecting Figure 2 that we must have $H \subseteq B$. We may find an arc $\beta \in \widehat{B}$ such that $\partial\beta = \partial[L]$ and $(\widehat{B}, \beta)$ is an unknotted ball-arc pair. It follows that $\beta \cup [L]$ is an unknotted loop (see [9] for a definition). We can consider $\beta \cup [L]$ as the union of the arcs $[K]$ and $\beta \cup [L \setminus K]$; but $(H, [K])$ is a *knotted* ball-arc pair, and $(\widehat{H}, \beta \cup [L \setminus K])$ is a ball-arc pair (because $L \cap H = K$). This contradicts a standard theorem which states that no knot has an additive inverse (Corollary 2.5 in [9]).

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