PERCOLATION GAMES, PROBABILISTIC CELLULAR AUTOMATA, AND THE HARD-CORE MODEL

ALEXANDER E. HOLROYD, IRÈNE MARCOVICI, AND JAMES B. MARTIN

ABSTRACT. Let each site of the square lattice $\mathbb{Z}^2$ be independently declared *closed* with probability p, and otherwise *open*. Consider the following game: a token starts at the origin, and the two players take turns to move it from its current site x to an open site in $\{x + (0, 1), x + (1, 0)\}$; if both these sites are closed, then the player to move loses the game. Is there positive probability that the game is *drawn* with best play – i.e. that neither player can force a win? This is equivalent to the question of ergodicity of a certain elementary one-dimensional probabilistic cellular automaton (PCA), which has been studied in the contexts of enumeration of directed animals, the golden-mean subshift, and the hard-core model. Ergodicity of the PCA has been noted as an open problem by several authors. We prove that the PCA is ergodic for all $0 < p < 1$, and correspondingly that the game on $\mathbb{Z}^2$ has no draws. We establish similar results for a certain misère variant of the game and a PCA associated to it.

On the other hand, we prove that the analogous game *does* exhibit draws for sufficiently small p on various directed graphs in higher dimensions, including an oriented version of the even sublattice of $\mathbb{Z}^d$ in all $d \geq 3$. This is proved via a dimension reduction to a hard-core lattice gas in dimension $d - 1$. We show that draws occur whenever the corresponding hard-core model has multiple Gibbs distributions. We conjecture that draws occur also on the standard oriented lattice $\mathbb{Z}^d$ for $d \geq 3$, but here our method encounters a fundamental obstacle.

1. INTRODUCTION

We introduce and study **percolation games** on various graphs. For the lattice $\mathbb{Z}^2$, we show that the probability of a draw is 0; this is equivalent to showing the ergodicity of a certain probabilistic cellular automaton. In higher dimensions, we prove that draws can occur, by developing a connection to the question of multiplicity of Gibbs distributions for the hard-core model.

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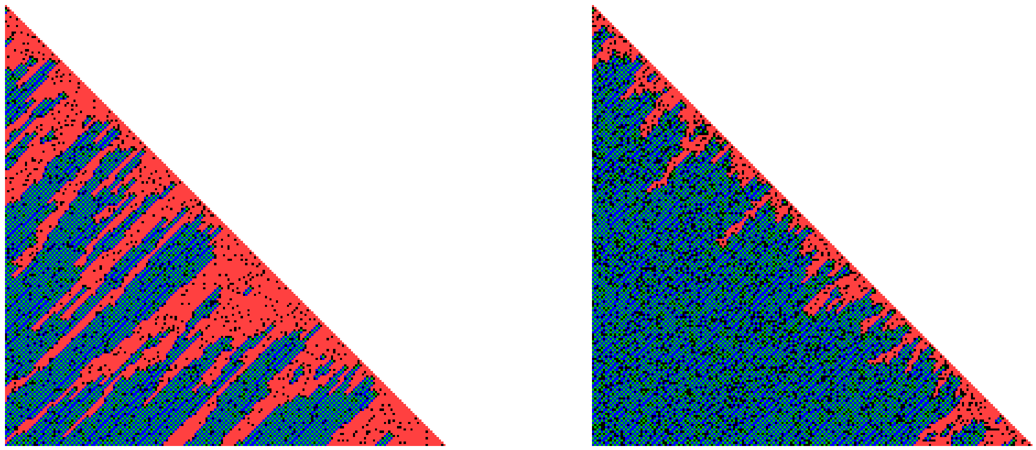

FIGURE 1. Outcomes of the percolation game on the finite region $\{x \in \mathbb{Z}_+^2 : x_1 + x_2 \leq 200\}$, declaring a draw if the token reaches the diagonal $x_1 + x_2 = 200$, and with $p = 0.1$ (left) and $p = 0.2$ (right). Colours indicate the outcome when the game is started from that site: first player win (blue); first player loss (green); draw (red). Closed sites are black.

1.1. **Two dimensional games and ergodicity.** Let each site of $\mathbb{Z}^2$ be closed with probability p and open with probability $1-p$, independently for different sites. Consider the following two-player game. A token starts at the origin. The players move alternately; if the token is currently at x, a move consists of moving it to $x + (0,1)$ or to $x + (1,0)$. The token is only allowed to move to an open site. In accordance with the "normal play rule" of combinatorial game theory, if both the sites $x + (0,1)$ and $x + (1,0)$ are closed, then the player whose turn it is to move loses the game. The entire configuration of open and closed sites is known to both players at all times. We call this the **percolation game** on $\mathbb{Z}^2$.

If $p \geq 1 - p_c$, where p_c is the critical probability for directed site percolation, then, with probability 1, only finitely many sites can be reached from the origin along directed paths of open sites, and so the game must end in finite time. In particular, one or other player must have a winning strategy. (A **strategy** for one or other player is a map that assigns a legal move, where one exists, to each vertex; a **winning** strategy is one that results in a win for that player, whatever strategy the other player uses.) Suppose on the other hand that $p < 1 - p_c$; is there now a positive probability that neither player has a winning strategy? In that case we say that the game is a **draw**, with the interpretation that it continues for ever with best play. (When $p = 0$ the game is clearly always a draw).

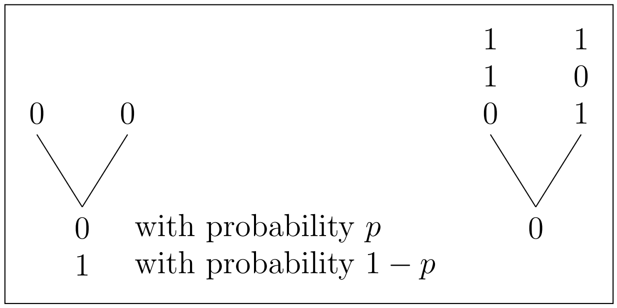

FIGURE 2. The probabilistic cellular automaton (PCA) A_p.

See Figure 1 for simulations on a finite triangular region, with draws imposed as a boundary condition. As the size of this region tends to ∞, the probability of a draw starting from the origin converges to the probability of a draw on $\mathbb{Z}^2$; the question is whether this limiting probability is positive for any p.

Related questions are considered in [HM] and [BHMW15], in which the underlying graph is respectively a Galton-Watson tree, and a random subset of the lattice with undirected moves.

In our case of a random subset of $\mathbb{Z}^2$ with directed moves, the outcome (win, loss, draw) of the game started from each site can be interpreted in terms of the evolution of a certain one-dimensional discrete-time probabilistic cellular automaton (PCA); the state of the PCA at a given time relates to the outcomes associated to the sites on a given Northwest-Southeast diagonal of $\mathbb{Z}^2$.

The PCA has alphabet $\{0,1\}$ and universe $\mathbb{Z}$, so that a configuration at a given time is an element of $\{0,1\}^{\mathbb{Z}}$. (The three game outcomes will correspond to the two states of the PCA via a coupling of two copies of the PCA). The evolution of the PCA is as follows. Given a configuration η_t at some time t, the configuration η_{t+1} at time $t+1$ is obtained by updating each site $n \in \mathbb{Z}$ simultaneously and independently, according to the following rule.

- If $\eta_t(n-1) = \eta_t(n) = 0$, then $\eta_{t+1}(n)$ is set to 0 with probability p and 1 with probability $1-p$.
- Otherwise (i.e. if at least one of $\eta_t(n-1)$ and $\eta_t(n)$ is 1), $\eta_{t+1}(n)$ is set to 0 with probability 1.

We denote this PCA A_p, and call it the **hard-core PCA**. Its evolution rule at each site is illustrated in Figure 2. (The time coordinate t increases from top to bottom, and the spatial coordinate n increases from left to right).

The hard-core PCA A_p has already been studied from a number of different perspectives. It is closely related to the enumeration of directed lattice animals, which are classical objects in combinatorics. The link was originally made by Dhar

[Dha83], and subsequent work includes [BM98, LBM07] – see also Section 4.2 of the survey of Mairesse and Marcovici [MM14a] for a short introduction. It also has strong connections to the hard-core lattice gas model in statistical physics (which also has applications for example to the modeling of communications networks) – see Section 3 of this paper. The case $p = 1/2$ in particular relates to the measure of maximal entropy of the golden mean subshift in dynamical systems – see [Elo96] and also [Mar13, Chapter 8].

Formally, we take A_p to be the operator on the set of distributions on $\{0,1\}^{\mathbb{Z}}$ representing the action of the PCA; if μ is the distribution of a configuration in $\{0,1\}^{\mathbb{Z}}$, then $A_p\mu$ is the distribution of the configuration obtained by performing one update step of the PCA. A **stationary distribution** of a PCA F is a distribution μ such that $F\mu = \mu$. (More generally, μ is k**-periodic** if $F^k\mu = \mu$, and **periodic** if it is k-periodic for some $k \geq 1$.) A PCA is said to be **ergodic** if it has a unique stationary distribution and if from any initial distribution, the iterates of the PCA converge to that stationary distribution (in the sense of convergence in distribution with respect to the product topology).

It is straightforward to show that if p is sufficiently large, then the hard-core PCA A_p is ergodic. The question of whether A_p is ergodic for *all* $p \in (0,1)$ has been mentioned as an open problem by several authors, for example in [TVS+90], [LBM07], [MM14a]. A link between the hard-core PCA A_p and the percolation game was already mentioned by [LBM07], and it is relatively easy to show that the percolation game has positive probability of a draw if and only if A_p is non-ergodic.

We also consider a certain "misère" variant of the game which we call the **target game**. In the percolation game described above, moves to closed sites were forbidden; this is of course equivalent to allowing all moves, but with the proviso that a player who moves to a closed site loses the game. In the target game, the game is instead *won* by the first player who moves to a closed site. In the same way that the percolation game corresponds to the hard-core PCA A_p, we show that the target game corresponds to another PCA, which we denote by B_p, whose update rule is shown in Figure 3.

PCA that are defined on $\mathbb{Z}$ and whose alphabet and neighbourhood are both of size 2 are sometimes called *elementary* PCA. A variety of tools have been developed to study their ergodicity. Under the additional assumption of left-right symmetry of the update rule, these PCA are defined by only three parameters: the probabilities to update a cell to state 1 if its neighbourhood is in state 00, 11, or 01 (which is the same as for 10). Existing methods can be used to handle more than 90% of the volume of the cube $[0,1]^3$ defined by this parameter space, but when p is small, the PCA A_p and B_p belong to an open domain of the cube where none of the previously known criteria hold [TVS+90, Chapter 7].

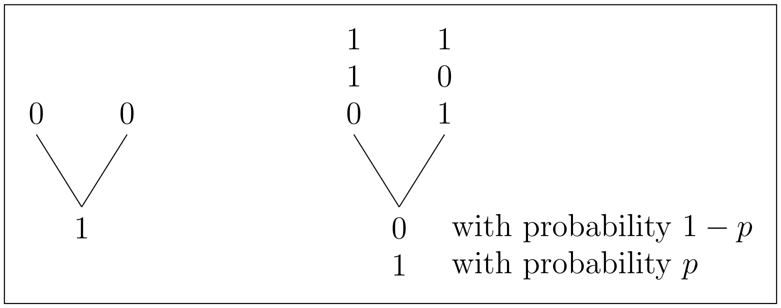

FIGURE 3. The PCA B_p.

We now state our first main result.

Theorem 1. *For any $p \in (0,1)$, the PCA A_p and B_p are both ergodic, and the probability of a draw is zero for both the percolation game and the target game on $\mathbb{Z}^2$.*

We prove ergodicity by considering the **envelope** PCAs corresponding to A_p and B_p, which are PCAs with an expanded alphabet $\{0, ?, 1\}$. The envelope PCA corresponds to the status of the game started from each site (with the symbols 0, ? and 1 corresponding to wins, draws and losses respectively). An evolution of the envelope PCA can be used to encode a coupling of two copies of the original PCA, with a ? symbol denoting sites where the two copies disagree. We introduce a new method involving a positive weight assigned to each ? symbol (whose value depends on the state of nearby sites). The correct choice of weights is delicate and non-obvious. We show that if the process is translation-invariant, then the average weight per site strictly decreases under the evolution of the envelope PCA, unless it is 0. It follows that any translation-invariant stationary distribution for the envelope PCA has no ? symbols, with probability 1, and from this we will be able to deduce that the game has no draws with probability 1, so that the original PCA is ergodic. Although the proof of ergodicity could be phrased so as not to refer to games, the notion is useful as a semantic tool and a guide to intuition.

Our proof of Theorem 1 follows almost identical steps for the PCA B_p (and the target game) as for A_p (and the percolation game). However, in some respects the two cases are very different. In particular, for the percolation game, combining Theorem 1 with known methods permits an explicit description of the distribution of game outcomes along a diagonal, as a Markov chain. Consequently, we show that the probability that the first player wins the percolation game or the origin is closed is

$$\frac{1}{2}\left(1+\sqrt{\frac{p}{4-3p}}\right). \tag{1.1}$$

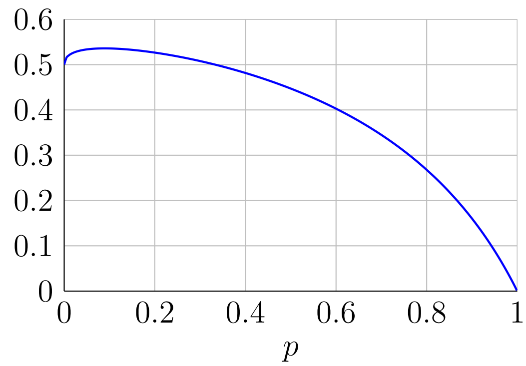

FIGURE 4. The probability that the first player wins the percolation game, conditional on the origin being open, as a function of the density p of closed sites.

See Figure 4 for the conditional first player win probability given that the origin is open. This conditional probability is greater than 1/2 if and only if $p \in (0, 1/3)$, and is maximized at $p = (2 - \sqrt{3})/3 = 0.0893...$. These methods seemingly do not work for the target game, and we do not know an explicit expression for the win probability. Furthermore, for an alternative misère version of the game (see Question 4.5 in the final section of the paper) there is apparently no similar connection to a PCA with alphabet $\{0, 1\}$, and we do not have a proof that the probability of a draw is 0. This is somewhat reminiscent of the situation for sums of combinatorial games [Con01], where the well-developed theory of "normal play" games extends only in very limited cases to their misère cousins.

1.2. **Higher dimensions and the hard-core model.** We also consider extensions of the percolation game, described above for $\mathbb{Z}^2$, to more general directed graphs, and in particular to lattices in higher dimensions. Theorem 1 tells us that in two dimensions, the probability of a draw is 0 for all positive p, but we find a very different picture in three and higher dimensions.

Let $G = (V, E)$ be a locally finite directed graph. For $x \in V$, let $\mathrm{Out}(x)$ and $\mathrm{In}(x)$ be the sets of out-neighbours and in-neighbours of x respectively. For the percolation game on G, let each vertex x be closed with probability p and open with probability $1-p$, independently for different vertices. A token starts at some vertex, and the two players move alternatively; if the token is currently at x, a move consists of moving it to any vertex in $\mathrm{Out}(x)$. The token is only allowed to move to open sites; if all the vertices in $\mathrm{Out}(x)$ are closed, then the player to move loses the game.

For graphs G with an appropriate structure, we develop a connection to the hard-core model on a related undirected graph in one fewer dimensions, to obtain a criterion under which the game is drawn with positive probability.

For an undirected graph with vertex set W, and any $\lambda > 0$, a **Gibbs distribution** for the **hard-core model** on W with **activity** λ is a probability distribution on configurations $\eta \in \{0,1\}^W$ such that

$$\mathbb{P}\Big(\eta(v) = 1 \;\Big|\; (\eta(w) : w \neq v)\Big) = \begin{cases} \dfrac{\lambda}{1+\lambda}, & \text{if } \eta(w) = 0 \text{ for all neighbours } w \text{ of } v; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Any such Gibbs distribution is concentrated on configurations η that correspond to independent sets, in the sense that no two neighbouring vertices v and w have $\eta(v) = \eta(w) = 1$. If W is finite, then there is a unique Gibbs distribution, which is the probability distribution that puts weight proportional to $\prod_{v \in W} \lambda^{\eta(v)}$ on each configuration η that is supported on an independent set. However, for infinite graphs, there may be multiple Gibbs distributions. A well-known example is the lattice $\mathbb{Z}^d$ with nearest-neighbour edges. For $d = 1$, there is a unique Gibbs distribution for all activities λ. However, for $d \geq 2$, there exist multiple Gibbs distributions when λ is sufficiently large [Dob65].

Returning to the percolation game on a directed graph G, we now give the key assumptions on G that are required for our dimension reduction method. Suppose there is a partition $(S_k : k \in \mathbb{Z})$ of the vertex set V of G, and an integer $m \geq 2$, such that the following conditions hold.

- (A1) For all $x \in S_k$, we have $\mathrm{Out}(x) \subset S_{k+1} \cup \cdots \cup S_{k+m-1}$.
- (A2) There is a graph automorphism ϕ of G that maps S_k to S_{k+m} for every k, and such that $\mathrm{Out}(x) = \mathrm{In}(\phi(x))$ for all x.

Then let D_k be the graph with vertex set $S_k \cup \cdots \cup S_{k+m-1}$, with an undirected edge (x, y) whenever (x, y) is a (directed) edge of V. (Below for convenience we will also use D_k to denote the vertex set $S_k \cup \cdots \cup S_{k+m-1}$.) It is straightforward to show that under conditions (A1) and (A2), the graphs D_k are isomorphic to each other for all $k \in \mathbb{Z}$ (see Lemma 3.1); write D for a generic graph which is isomorphic to any of the D_k. Then we have the following criterion for positive probability of draws.

Theorem 2. *Suppose that the directed graph G satisfies (A1) and (A2). If there exist multiple Gibbs distributions for the hard-core model on D with activity λ, then the percolation game on G with $p = 1/(1 + \lambda)$ has positive probability of a draw from some vertex.*

The simplest case in which to understand the conditions (A1) and (A2) is when G is the directed lattice $\mathbb{Z}^2$, with $\mathrm{Out}(x) = \{x + (1, 0), x + (0, 1)\}$ (the setting of Theorem 1 in Section 1.1). Then we may take the partition of $\mathbb{Z}^2$ into Northeast-Southwest diagonals given by $S_k := \{(x_1, x_2) : x_1 + x_2 = k\}$, along with the bijection $\phi(x) = x + (1, 1)$, and $m = 2$. The graph D then consists of the vertices of two successive diagonals, and is thus isomorphic to the line $\mathbb{Z}$. (In the context of PCA, D is sometimes called the *doubling graph.*)

Since, as noted above, there is a unique Gibbs distribution for the hard-core model on $\mathbb{Z}$ for all $\lambda > 0$, Theorem 2 does not imply the existence of draws for any $p \in (0, 1)$. (Indeed, that would contradict Theorem 1).

In higher dimensions the picture is different. We will give several examples of relevant graphs in Section 3.2 and Theorem 3 below. For the current discussion, consider the case where G has vertex set $\mathbb{Z}^d_{\text{even}} := \{x \in \mathbb{Z}^d : \sum x_i \text{ is even}\}$, with directed edges given by $\mathrm{Out}(x) := \{x \pm e_i + e_d : 1 \le i \le d-1\}$ (where e_i is the ith standard basis vector in $\mathbb{Z}^d$). So $\mathrm{Out}(x)$ has size $2(d-1)$; any move of the game increases the dth coordinate by 1 and also changes exactly one of the other coordinates by 1 in either direction. In two dimensions, this game is isomorphic to the original game on $\mathbb{Z}^2$. For general d, conditions (A1) and (A2) hold with $m = 2$ if we set $S_k = \{x \in \mathbb{Z}^d_{\text{even}} : x_d = k\}$ and $\phi(x) = x + 2e_d$. One then finds that D is isomorphic to the standard $(d-1)$-dimensional cubic lattice $\mathbb{Z}^{d-1}$ with nearest-neighbour edges. As mentioned above, there are multiple Gibbs distributions for the hard-core model on $\mathbb{Z}^{d-1}$ whenever $d \ge 3$ and λ is large enough; then Theorem 2 tells us that the percolation game on G has positive probability of a draw when p is sufficiently small. We do not know whether the draw probability is monotone in p, nor even whether it is supported on a single interval (giving a single critical point).

To prove Theorem 2, we consider a recursion, analogous to the earlier PCA, expressing game outcomes starting from vertices in S_k in terms of outcomes starting in $S_{k+1} \cup \cdots \cup S_{k+m}$. Via the graph isomorphism from D_k to D, the iteration of this recursion can be reinterpreted as a version of Glauber dynamics for the hard-core model on D. If the hard-core model has multiple Gibbs distributions, then they correspond to multiple stationary distributions for the recursion on G, and from this we will deduce that draws occur.

The stationary distributions for the recursion on G that arise from the above correspondence have a certain special property, which in the case $m = 2$ can be viewed as a version of time-reversibility. In cases where the hard-core model has a *unique* Gibbs distribution (such as when $G = \mathbb{Z}^2$ and $D = \mathbb{Z}$), the argument implies that the recursion has only one stationary distribution with this reversibility property, but says nothing about the possibility of other stationary distributions.

It is for this reason that the implication in Theorem 2 is in only one direction, and that the proof of Theorem 1 requires a different argument.

Unfortunately, the following very natural example is *not* amenable to our methods. Let G be the standard cubic lattice $\mathbb{Z}^d$ with edge orientations given by $\mathrm{Out}(x) = \{x + e_i : 1 \leq i \leq d\}$. Theorem 2 does not apply for $d \geq 3$, because there is no choice of m and the automorphism ϕ such that (A2) holds. We conjecture that, nonetheless, the percolation game has positive probability of a draw whenever p is sufficiently small.

1.3. **Further background.** The celebrated *positive rates conjecture* is the assertion that in one dimension, any finite-state finite-range PCA is ergodic, provided the transition probability to any state given any neighbourhood states is positive (the latter is the "positive rates" condition). This contrasts with two and higher dimensions, where for example the low temperature Ising model is well-known to be non-ergodic. Despite persuasive heuristic arguments in favour of the positive rates conjecture, Gács [Gác01] has given an extremely complicated one-dimensional PCA refuting it. (See also [Gra01].) However, it is still natural to hypothesize that all "sufficiently simple" one-dimensional PCA with positive rates are ergodic.

The PCA A_p and B_p are very simple, but do not satisfy the positive rates condition. (E.g. in A_p, the word 11 deterministically yields 0.). However, similar but weaker conditions do hold; for example, any finite word in $\{0,1\}^n$ has positive probability of yielding any word in $\{0,1\}^{n-2}$ after *two* steps of the evolution. In light of this and the above remarks, it would have been very surprising if these PCA were not ergodic. Nonetheless, *proving* ergodicity is often very difficult, even in cases where it appears clear from heuristics or simulations.

Another case in point is the notorious *noisy majority* model on $\mathbb{Z}^d$. Here, a configuration is an element of $\{0,1\}^{\mathbb{Z}^d}$. The update rule is that with probability $1-p$, a site adopts the more popular value in $\{0,1\}$ among itself and its $2d$ neighbours; with probability p it adopts the other value. In dimensions $d \geq 2$ it is expected that this PCA should behave similarly to the Ising model: it should be ergodic for p sufficiently close to $1/2$, and non-ergodic for p sufficiently small, with a unique critical point separating the two regimes. However, proving any of this appears very challenging. See e.g. [BBJW10, Gra01] and the references therein for more information. One key difficulty with the noisy majority model is the lack of reversibility of the dynamics (in contrast to the Glauber dynamics for the Ising model, for example). This can be compared to the difficulty of obtaining a result like Theorem 2 in the absence of conditions such as (A1) and (A2); see the discussion above at the end of Subsection 1.2.

In a different direction, a variant of the notion of envelope cellular automata has recently been combined with percolation ideas in [GH15], to prove the surprising fact that certain deterministic one-dimensional cellular automata exhibit order from *typical* finitely supported initial conditions, but disorder from exceptional initial conditions.

1.4. **Organization of the paper.** In Section 2 we explain the link between the PCA A_p and B_p and the percolation game and target game respectively in $\mathbb{Z}^2$. We also establish several basic results concerning monotonicity and ergodicity. The local weighting on configurations is introduced in Subsection 2.3, and the proof of ergodicity is then given in Subsection 2.4.

The relation to the hard-core model is then developed in Section 3. We start by considering the case of $\mathbb{Z}^2$ where the ideas are simplest, and in particular we will derive the formula (1.1) for the winning probability. The general case is then treated in Subsection 3.2, where Theorem 2 is proved. In Subsection 3.4 and Theorem 3, we give a variety of examples of the application of Theorem 2 to graphs with vertex set $\mathbb{Z}^d$ for $d \geq 3$, for which the role of the doubling graph D is played by various lattice structures. We also give an extension of Theorem 2 in Proposition 3.1 in Subsection 3.5, using a variant form of the correspondence to the hard-core model.

We conclude in Section 4 with some open problems.

2. Percolation games and probabilistic cellular automata

2.1. **The PCA for the percolation game.** Consider the percolation game on $\mathbb{Z}^2$ as defined in the introduction.

Suppose x is an open site of $\mathbb{Z}^2$. Let $\eta(x)$ be W, L or D according to whether the game started with the token at x is win for the first player, a loss for the first player, or a draw, respectively. (Recall that we assume optimal play, with the players able to see entire configuration of open and closed sites when deciding on their strategies). If x is a closed site, it is convenient to set $\eta(x) = \mathrm{W}$. (We can imagine that a player is allowed to move the token to x, but with the effect that the game is then declared an immediate win for the opponent).

Recall that $\mathrm{Out}(x) = \{x + e_1, x + e_2\}$ is the set of sites to which the token can move from x. By considering the first move, we have the following recursion for the status of the sites:

$$x \text{ closed } \Rightarrow \eta(x) = \mathrm{W};$$

$$x \text{ open } \Rightarrow \eta(x) = \begin{cases} \mathrm{L} & \text{if } \eta(y) = \mathrm{W} \text{ for all } y \in \mathrm{Out}(x) \\ \mathrm{W} & \text{if } \eta(y) = \mathrm{L} \text{ for some } y \in \mathrm{Out}(x) \\ \mathrm{D} & \text{otherwise.} \end{cases} \tag{2.1}$$

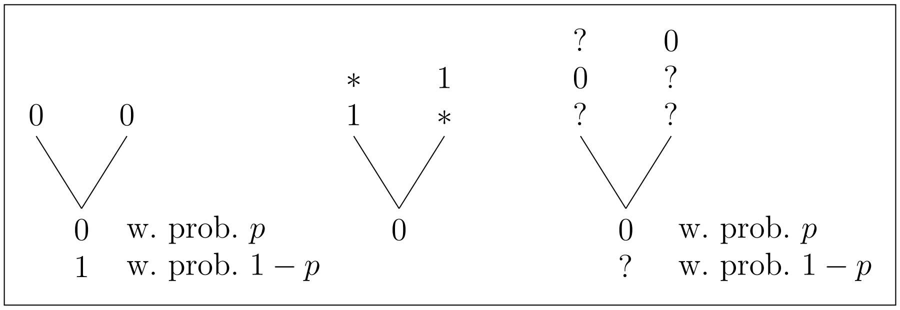

Figure 5. The PCA F_p ($*$ denotes an arbitrary symbol).

For $k \in \mathbb{Z}$, let S_k be the set $\{x = (x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 = k\}$, a NW-SE diagonal of $\mathbb{Z}^2$. The recursion (2.1) gives us the values $(\eta(x) : x \in S_k)$ in terms of the values $(\eta(x) : x \in S_{k+1})$ together with the information about which sites in S_k are closed.

It is important to note that it is not *a priori* clear whether the recursion (2.1) suffices to determine η uniquely from the configuration of open and closed sites. Indeed, in the trivial case $p = 0$ when all sites are open, we have $\eta(x) = \mathrm{D}$ for all x, but (2.1) has other solutions: one is to set $\eta'(x)$ equal to L or W according to whether $x_1 + x_2$ is odd or even. Such considerations are in fact central to many of our arguments. One way to interpret our main result, Theorem 1, is as saying that (2.1) does have a unique solution almost surely for all $0 < p < 1$ (and similarly for the target game). In contrast, for the higher dimensional variants considered later, the analogous recursions admit multiple solutions for suitable p.

Via (2.1), we can regard the configurations on successive diagonals S_k, as k decreases, as successive states of a one-dimensional PCA. Let us introduce the following recoding:

$$\mathrm{W} = 0, \quad \mathrm{L} = 1, \quad \mathrm{D} = ?.$$

(In the coupling arguments below, the symbol ? will be interpreted as marking a site at which the value is "unknown". The choice to assign W = 0 and L = 1, rather than the other way around, say, will be important for the later connection with hard-core models.) The PCA evolves as follows: given the values for sites in S_{k+1}, each value $\eta(x)$ for $x \in S_k$ is derived independently using the values $\eta(x+e_1)$ and $\eta(x+e_2)$, according to the scheme given in Figure 5 (where a $*$ represents an arbitrary symbol in $\{0, ?, 1\}$).

We denote the corresponding PCA F_p. Although we have defined it as a process in the plane, we can also regard it as a PCA on $\mathbb{Z}$ with a configuration in $\{0, ?, 1\}^{\mathbb{Z}}$ evolving in time by setting

$$\eta_t(n) = \eta\big((-t-n, n)\big). \tag{2.2}$$

(Here we have made the arbitrary choice to offset leftward as time increases,so that the PCA rule gives $\eta_{t+1}(n)$ in terms of $\eta_t(n)$ and $\eta_t(n+1)$.) As in Section 1.1, formally we take F_p to be an operator on the set of distributions on $\{0,?,1\}^{\mathbb{Z}}$ representing the action of the PCA.

In the setting of the percolation game, translation invariance of the whole process on $\mathbb{Z}^2$ implies that the distribution of the configuration on the diagonal S_k does not depend on k; that is, the distribution of $\big(\eta((k-n,n)) : n \in \mathbb{Z}\big)$ does not depend on k and is a stationary distribution of F_p. In addition, this distribution is itself invariant under the action of translations of $\mathbb{Z}$.

We next note two useful monotonicity properties for the PCA F_p. In terms of the game, they have natural interpretations: (i) an advantage for one player translates to a disadvantage for the other; and (ii) declaring draws at some positions can only result in more draws elsewhere.

Lemma 2.1. *Let μ and ν be probability distributions on $\{0,?,1\}^{Z}$.*

- (i) *If $\mu \leq \nu$, where $\leq$ denotes stochastic domination with respect to the coordinatewise partial order induced by $0 < ? < 1$, then $F_p\mu \geq F_p\nu$. (Note the reversal of the inequality).*
- (ii) *If $\mu \trianglelefteq \nu$, where $\trianglelefteq$ denotes stochastic domination with respect to the coordinatewise partial order induced by $0 \triangleleft ? \triangleright 1$, then $F_p\mu \trianglelefteq F_p\nu$.*

Proof. We can use the recursion (2.1) to give a coupling of a single step of the PCA F_p started from two different configurations. Suppose we fix values $(\eta(x) : x \in S_{k+1})$ and $(\widetilde{\eta}(x) : x \in S_{k+1})$, in such a way that $\eta(x) \leq \widetilde{\eta}(x)$ for all $x \in S_{k+1}$ (where $\leq$ is the coordinatewise order on configurations induced by $0 < ? < 1$). Now use (2.1) to obtain values $\eta(x)$ and $\widetilde{\eta}(x)$ for $x \in S_k$, using the same realization of closed and open sites in S_k in each case. It is straightforward to check that in that case $\eta(x) \geq \widetilde{\eta}(x)$ for each $x \in S_k$. Hence the operator F_p is decreasing in the desired sense.

Similarly, if $\eta(x) \trianglelefteq \widetilde{\eta}(x)$ for all $x \in S_{k+1}$, then we obtain $\eta(x) \trianglelefteq \widetilde{\eta}(x)$ also for each $x \in S_k$. So in this case the operator F_p is increasing as desired. ☐

If we restrict the PCA F_p to configurations that do not contain the symbol ?, we recover precisely the hard-core PCA A_p defined in the introduction. In the terminology of Bušić et al. [BMM13], the PCA F_p is the **envelope** PCA of A_p. A copy of the PCA F_p can be used to represent a coupling of two or more copies of the PCA A_p, started from different initial conditions. The symbol ? represents a site whose value is not known, i.e. one which may differ between the different copies.

Specifically, consider starting copies of the hard-core PCA A_p from several different initial conditions, represented by configurations on the diagonal S_k for some fixed k. As in the proof of Lemma 2.1, a natural coupling is provided by the recursion (2.1), using the same realization of the closed and open sites on $(S_r : r < k)$. In particular, let $k > 0$ and consider three copies η, $\widetilde{\eta}$ and $\eta^?$, with η and $\widetilde{\eta}$ started from arbitrary initial conditions on S_k, while $\eta^?(x) = ?$ for all $x \in S_k$. (So that $\eta^?$ is maximal for the ordering $\trianglelefteq$ in Lemma 2.1(ii)). Then we have that $\eta(x) \trianglelefteq \eta^?(x)$ and $\widetilde{\eta}(x) \trianglelefteq \eta^?(x)$ for all $x \in S_r$ with $r < k$. This implies that if $\eta(x) \neq \widetilde{\eta}(x)$, then $\eta^?(x) = ?$.

In terms of the game, we have the following interpretation: if the origin $O = (0, 0)$ is an open site, and $\eta^?(O) = 0$ (respectively $\eta^?(O) = 1$) then the first (respectively second) player can force a win within at most k moves of the game.

The ergodicity of an envelope PCA implies the ergodicity of the original PCA, but the converse is not true in general. In our case, however, we can use the monotonicity property in Lemma 2.1(i) to show that the two are equivalent.

Proposition 2.1. *The PCA F_p is ergodic if and only if A_p is ergodic.*

Proof. It is clear from the definitions that if F_p is ergodic, then A_p is also ergodic. Conversely, suppose that A_p is ergodic. Let μ be a distribution on $\{0, ?, 1\}^{\mathbb{Z}}$, and let δ_0 and δ_1 the distributions concentrated on the configurations "all 0s" and "all 1s". Then $\delta_0 \leq \mu \leq \delta_1$, so by Lemma 2.1(i), for $k \geq 0$ we have either $F_p^k\delta_0 \leq F_p^k\mu \leq F_p^k\delta_1$ or $F_p^k\delta_0 \geq F_p^k\mu \geq F_p^k\delta_1$, according to whether k is even or odd. But $F_p^k\delta_0 = A_p^k\delta_0$ and $F_p^k\delta_1 = A_p^k\delta_1$, and by ergodicity of A_p, the latter two sequences converge as $k \to \infty$ to the same distribution π, so $F_p^k\mu$ also converges to π. Thus F_p is also ergodic. □

Proposition 2.2. *For each $p \in [0, 1]$, the percolation game has probability 0 of a draw if and only if A_p is ergodic.*

Proof. If A_p is ergodic then so is F_p, and so the unique invariant distribution of F_p has no ? symbols. But we know that the distribution of the game outcomes along a diagonal S_k is invariant for F_p. Hence with probability 1, there are no sites from which the game is drawn.

For the converse, let ω be a random configuration of open and closed sites on $\mathbb{Z}^2$ chosen according to the percolation measure. Consider any site $x \in S_0$. If the game started from x is not a draw, then (since at each turn the player to move has only finitely many options) one player has a strategy that guarantees a win in fewer than N moves, where $N \in \mathbb{N}$ is a finite random variable that depends on ω. Consequently, if we assign any configuration of states $0, ?, 1$ to S_N and compute the resulting states on $(S_n : 0 \leq n < N)$ using the recursion (2.1) and

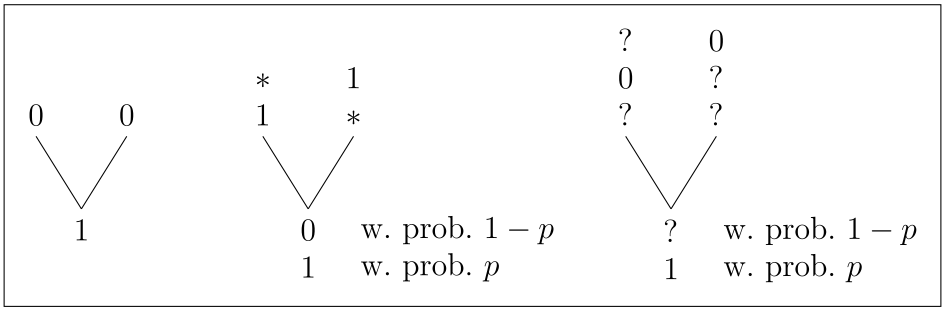

Figure 6. The PCA G_p.

the configuration ω of open and closed sites, the resulting state at x is the same as its state for the percolation game on $\mathbb{Z}^2$ with configuration ω.

Let γ be the random configuration of game outcomes on S_0 arising from ω. Also, fix a distribution ν on $\{0,1\}^{\mathbb{Z}}$, and let γ_n be the configuration on S_0 that results from assigning a configuration with law ν to S_n, independent of ω, and applying (2.1) as described above. By the argument in the previous paragraph, if the probability of a draw is 0, then γ_n converges almost surely to γ (in the product topology). Hence also the distribution of γ_n converges to that of γ. But γ_n has distribution $A_p^n\nu$, so $A_p^n\nu$ converges as $n\to\infty$ to the distribution of γ, which does not depend on ν. Hence A_p is ergodic. □

2.2. **The target game and the corresponding PCA.** We now turn to the target game, in which the winner is the first player to move to a closed site. As before, we can introduce a PCA to describe the status of the positions. The recursion (2.1) still applies when x is open, but now if x is closed we set instead $\eta(x) = \mathrm{L} = 1$. The equivalents of A_p and F_p for the target game are the PCA B_p and G_p defined via Figures 3 and 6.

We may see the PCA B_p as a composition of Stavskaya's PCA and the flip operator. Stavskaya's PCA (see for example [TVS$^+$90]) is given by the local rule which sets $\eta_t(n) = 0$ with probability p and otherwise $\eta_t(n) = \max\{\eta_{t-1}(n), \eta_{t-1}(n-1)\}$, and provides a non-trivial example of a one-dimensional PCA where ergodicity fails. However, the ergodicity of the PCA B_p has not been studied previously. As in the case of A_p, it does not satisfy any of the previously known criteria for ergodicity.

Using exactly the same arguments as for Proposition 2.1 and Proposition 2.2, we have the following results.

Proposition 2.3. *The PCA G_p is ergodic if and only if B_p is ergodic.*

Proposition 2.4. *The target game has probability* 0 *of a draw if and only if B_p is ergodic.*

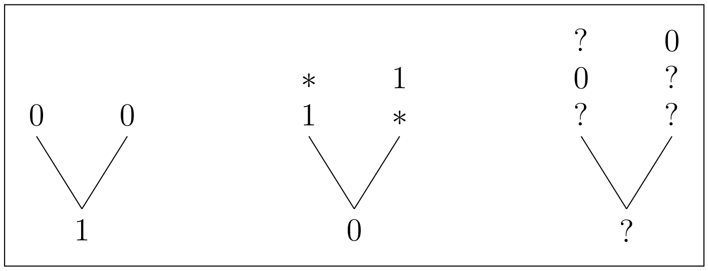

Figure 7. The deterministic cellular automaton D.

Below we will show that, for all $p > 0$, any stationary distribution for F_p or for G_p is concentrated on $\{0, 1\}^{\mathbb{Z}}$, so that the probability of a ? symbol is 0. Using Propositions 2.1 and 2.2, or 2.3 and 2.4 respectively, we will then be able to deduce the ergodicity of A_p and B_p as required for Theorem 1.

2.3. **The weight function.** We are concerned with the two PCA A_p and B_p on the alphabet $\{0, 1\}$, shown in Figures 2 and 3, and their envelope PCA F_p and G_p, shown in Figures 5 and 6.

We introduce a deterministic cellular automaton D defined in Figure 7. (Note that $D = F_0 = G_0$).

We also introduce the randomization operator R_p^0 on $\{0, ?, 1\}^{\mathbb{Z}}$ that changes each symbol into a 0 with probability p (making no change if it was already a 0), independently for different sites, and similarly the operator R_p^1 that changes each symbol into a 1 with probability p, independently for different sites. Observe that

$$F_p = R_p^0 \circ D \quad \text{and} \quad G_p = R_p^1 \circ D.$$

We now establish a property of the deterministic operator D. For a given configuration in $\{0, ?, 1\}^{\mathbb{Z}}$, let us weight the occurrences of the symbol ? as follows:

- if a ? is followed by a 01, then it receives weight 3;
- if a ? is followed by a 0 and then by something other than a 1, it receives weight 2;
- otherwise, a ? receives weight 1.

We say that the distribution of a configuration $\eta = (\eta_i : i \in \mathbb{Z})$ is **shift-invariant** if η and $(\eta_{i+k} : i \in \mathbb{Z})$ have the same distribution for each $k \in \mathbb{Z}$, and **reflection-invariant** if η and $(\eta_{-i} : i \in \mathbb{Z})$ have the same distribution. If μ is a distribution and $w \in \{0, ?, 1\}^n$ is a finite word, we write $\mu(w) := \mu\{\eta : (\eta_1, \ldots, \eta_n) = w\}$ for the corresponding cylinder probability. For a shift-invariant distribution μ on $\{0, ?, 1\}^{\mathbb{Z}}$, we introduce the quantity:

$$\mu(?01) + \mu(?0) + \mu(?),$$

which is the expected weight per site under μ.

Lemma 2.2. *If μ is a shift-invariant and reflection-invariant distribution on $\{0,?,1\}^{\mathbb{Z}}$, then*

$$D\mu(?01) + D\mu(?0) + D\mu(?) \leq \mu(?01) + \mu(?0) + \mu(?).$$

Proof. By looking at the possible pre-images of each pattern, we obtain the following three equalities:

$$\begin{aligned} D\mu(?) &= \mu(??) + \mu(0?) + \mu(?0), \\ D\mu(?0) &= \mu(??1) + \mu(0?1) + \mu(?01), \\ D\mu(?01) &= 0. \end{aligned}$$

Summing, and using reflection invariance to deduce $\mu(0?) = \mu(?0)$, we obtain

$$\begin{aligned} D\mu(?01) + D\mu(?0) + D\mu(?) - \mu(?01) - \mu(?0) & \\ &= \mu(??) + \mu(0?) + \mu(??1) + \mu(0?1) \\ &\leq \mu(??) + \mu(0?) + \mu(?1) \\ &= \mu(??) + \mu(?0) + \mu(?1) \\ &= \mu(?). \end{aligned} \tag{2.3}$$

□

Here is an informal way to explain the above result, which provides some insight into the reasons for the choice of weight. Let us consider a symmetric version of the weight system that we have introduced: for each symbol ?, we add its right-weight, as introduced above, to its left-weight, which is equal to 3 if it the previous letter is a 0 and if there is a 1 before it (pattern 10?), to 2 if the previous letter is a 0 and if there is something else than a 1 before, and to 1 otherwise. (Since in Lemma 2.2 we consider only reflection-invariant distributions, working with the symmetric weight is equivalent to working with the original weight).

Thus, the weight of the symbol ? in the pattern 1?1 is equal to $1 + 1 = 2$, while in the pattern 10??1, the weight of the first ? symbol is 3 (left) + 1 (right) = 4, and the weight of the second one is equal to $1 + 1 = 2$.

Figure 8 shows an example of evolution of the deterministic CA D from an initial configuration represented at the top (with time going down the page). The symmetrized weights of the symbols ? appearing in the space-time diagram are shown in red. As illustrated in the figure, from a pattern 1?1, the symbol ? disappears and the weight thus decreases, but in other cases the total weight is locally preserved.

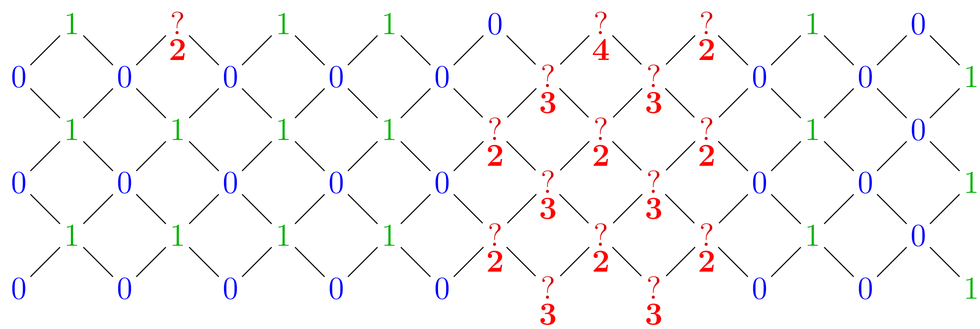

FIGURE 8. Example of evolution of the weight of a configuration under the operator D. Time runs down the page, and the weight of each ? symbol is given below it.

2.4. Proof of ergodicity.

Proposition 2.5. *For $0 < p < 1$, the PCA F_p has no stationary distribution in which the symbol ? appears with positive probability.*

Proof. It suffices to show that there is no shift-invariant and reflection-symmetric stationary distribution in which the symbol ? appears with positive probability. For consider iterating the PCA starting from the distribution $\delta_?$ concentrated on the configuration with ? at all sites. By Lemma 2.1(ii), the probability $F_p^n\delta_?(?)$ is non-increasing, and if there is any stationary distribution μ with positive probability of ?, then $F_p^n\delta_?(?)$ is bounded below by $\mu(?)$ for all n, and so does not converge to 0. Then any limit point of the sequence of Césaro sums of $F_p^n\delta_?$ is a stationary distribution that has positive probability of ?, and that is also shift-invariant and reflection-symmetric.

For a shift-invariant distribution ν on $\{0,?,1\}^{\mathbb{Z}}$, we have the following three equalities:

$$\begin{aligned} R_p^0\nu(?) &= (1-p)\nu(?), \\ R_p^0\nu(?0) &= p(1-p)\,\nu(?) + (1-p)^2\,\nu(?0) \\ R_p^0\nu(?01) &= p(1-p)^2\,\nu(?{*}1) + (1-p)^3\,\nu(?01). \end{aligned}$$

Here $*$ represents an unspecified symbol to be summed over, so that $\nu(?{*}1) := \sum_{a=0,?,1}\nu(?a1)$. Adding together these three equalities, we get

$$\begin{aligned} R_p^0\nu(?) + R_p^0\nu(?0) + R_p^0\nu(?01) &= \nu(?) + \nu(?0) + \nu(?01) + p(1-p)^2\,\nu(?{*}1) \\ &\quad - p^2\,\nu(?) - p(2-p)\,\nu(?0) - p(3-3p+p^2)\,\nu(?01). \end{aligned} \tag{2.4}$$

Let μ be a stationary distribution of F_p that is shift-invariant and reflection-symmetric, and set $\nu = D\mu$. Then, since $F_p = R_p \circ D$, we have $R_p^0\nu = \mu$. Hence by Lemma 2.2,

$$\nu(?01) + \nu(?0) + \nu(?) \le R_p^0\nu(?01) + R_p^0\nu(?0) + R_p^0\nu(?).$$

Then, since $p > 0$, it follows from (2.4) that

$$\begin{aligned}(1-p)^2\nu(?{*}1) &\ge p\nu(?) + (2-p)\nu(?0) + (3-3p+p^2)\nu(?01)\\ &\ge p\nu(?) + (2-p)\nu(?0).\end{aligned} \tag{2.5}$$

We now proceed to obtain bounds for the left and right sides of (2.5) in terms of cylinder probabilities of μ. We have $\mu(?) = R_p^0\nu(?) = (1-p)\nu(?)$. Also $\nu(?0) = D\mu(?0) = \mu(?01) + \mu(?1) - \mu(1?1) = \mu(?01) + \mu(?1)$, since $F_p\mu = \mu$ and the pattern $1?1$ has no preimage by F_p. We thus obtain:

$$\begin{aligned}p\nu(?) + (2-p)\nu(?0) &= \frac{p}{1-p}\mu(?) + (2-p)\big[\mu(?01) + \mu(?1)\big]\\ &= p\mu(?) + (2-p)\big[\mu(?01) + \mu(?1)\big] + \frac{p^2}{1-p}\,\mu(?)\\ &\ge p\mu(?1) + (2-p)\big[\mu(?01) + \mu(?1)\big] + \frac{p^2}{1-p}\,\mu(?)\\ &\ge 2\mu(?1) + \mu(?01) + \frac{p^2}{1-p}\,\mu(?).\end{aligned} \tag{2.6}$$

But we have

$$\begin{aligned}(1-p)^2\nu(?{*}1) &= R_p^0\nu(?{*}1)\\ &= \mu(?{*}1)\\ &= \mu(?01) + \mu(??1) + \mu(?11)\\ &\le \mu(?01) + 2\mu(?1).\end{aligned} \tag{2.7}$$

Putting together (2.5), (2.6) and (2.7) we obtain that $\mu(?) = 0$ as required. □

We now turn to the PCA corresponding to the target game.

Proposition 2.6. *For $0 < p < 1$, the PCA G_p has no stationary distribution in which the symbol $?$ appears with positive probability.*

Proof. As in the proof of Proposition 2.5 we need only consider stationary distributions that are shift-invariant and reflection-symmetric. For a shift-invariant distribution ν on $\{0,?,1\}^{\mathbb{Z}}$, we have the following equalities:

$$\begin{aligned} R_p^1\nu(?) &= (1-p)\nu(?) \\ R_p^1\nu(?0) &= (1-p)^2\nu(?0) \\ R_p^1\nu(?01) &= p(1-p)^2\nu(?0) + (1-p)^3\nu(?01). \end{aligned}$$

Thus,

$$\begin{aligned} R_p^1\nu(?) + R_p^1\nu(?0) + R_p^1\nu(?01) = \nu(?) + \nu(?0) + \nu(?01) + p(1-p)^2\nu(?0) \\ - p\nu(?) - p(2-p)\nu(?0) - p(3-3p+p^2)\nu(?01). \end{aligned}$$

Let μ be a shift-invariant, reflection-invariant, invariant distribution of G_p, and let $\nu = D\mu$. Then, $R_p^1\nu = \mu$. By Lemma 2.2,

$$\nu(?01) + \nu(?0) + \nu(?) \leq R_p^1\nu(?01) + R_p^1\nu(?0) + R_p^1\nu(?).$$

It follows that

$$\begin{aligned} (1-p)^2\nu(?0) &\geq \nu(?) + (2-p)\nu(?0) + (3-3p+p^2)\nu(?01) \\ &\geq (2-p)\nu(?0), \end{aligned}$$

which is possible only if $\nu(?0) = 0$. We then obtain $\mu(?0) = 0$, and it follows easily that $\mu(?) = 0$. □

Now we can quickly deduce our main result.

Proof of Theorem 1. We know that the distribution of the states (win, loss, draw) of the sites along a diagonal S_k in the percolation game is a stationary distribution for F_p. Since by Proposition 2.5, F_p has no stationary distribution with positive probability of ? for any $p > 0$, the probability of a draw in the percolation game must be 0. Then by Proposition 2.2, the PCA A_p is ergodic for each $p > 0$.

An identical argument applies for the target game, and the PCA B_p, using Propositions 2.6 and 2.4. □

3. Percolation games and the hard-core model

3.1. **The two-dimensional case.** In this section we develop the relationship between the percolation game and the hard-core model. We start in the setting of $\mathbb{Z}^2$ where the ideas are easiest to understand, but our main application will be in Section 3.2, when we establish a more general framework, and apply it to show that certain higher-dimensional games have positive probability of a draw when p is sufficiently small.

Consider the hard-core PCA A_p. This PCA is known to belong to a family of one-dimensional PCA having a stationary distribution that is itself a stationary

Markov chain indexed by $\mathbb{Z}$ [BGM69, TVS+90, MM14b]. This distribution, μ_p say, is the law of the stationary Markov chain on $\mathbb{Z}$ with transition matrix

$$P = \begin{pmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{pmatrix} = \begin{pmatrix} \frac{2-p-\sqrt{p(4-3p)}}{2(1-p)^2} & \frac{2p^2-3p+\sqrt{p(4-3p)}}{2(1-p)^2} \\ \frac{-p+\sqrt{p(4-3p)}}{2(1-p)} & \frac{2-p-\sqrt{p(4-3p)}}{2(1-p)} \end{pmatrix}, \tag{3.1}$$

on state space $\{0,1\}$. (See Section 4.2 of [MM14a] – note that p there corresponds to our $1-p$). In fact, the evolution of the PCA started from μ_p is time-reversible – the distribution of the two-dimensional space-time diagram obtained (via the correspondence at (2.2)) is invariant under reflection in the line $x_1 + x_2 = k$ for any k. (In addition, the distribution μ_p is itself reversible as a Markov chain on $\mathbb{Z}$, which corresponds to symmetry of the two-dimensional picture under reflection in the line $x_1 = x_2$).

By Theorem 1, we know that μ_p is in fact the unique stationary distribution of F_p, and therefore the probability that the first player wins the percolation game starting from the origin is, as claimed in (1.1),

$$\mu_p(0) = \frac{p_{1,0}}{p_{1,0} + p_{0,1}} = \frac{1}{2}\left(1 + \sqrt{\frac{p}{4-3p}}\right).$$

(The conditional probability of a win given that the origin is open is then given by $(\mu_p(0) - p)/(1-p)$.)

An illuminating way to understand the presence of this Markovian reversible stationary distribution is to consider the *doubling graph* of the PCA, corresponding to two consecutive times of its evolution [Vas78, KV80, TVS+90]. This is an undirected bipartite graph, connecting sites between which there is an influence induced by the rules of the PCA.

As in Section 2.1, we can think of a configuration of the PCA as indexed by a diagonal $S_k = \{(x_1, x_2) : x_1 + x_2 = k\}$ of $\mathbb{Z}^2$. A time-step of the PCA then corresponds to moving from a configuration on S_{k+1} to a configuration on S_k.

As before, let $\mathrm{Out}(x) = \{x + e_1, x + e_2\}$ for $x \in S_k$. The elements of $\mathrm{Out}(x)$ lie in S_{k+1}, and are the sites to which the token may move from sites x; they are the sites whose values appear on the right side of the recurrence (2.1) for the value $\eta(x)$. Then the bijection $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$\phi(x) = x + e_1 + e_2, \tag{3.2}$$

which maps S_k to S_{k+2} for each k, has the following symmetry property: for all x and y,

$$y \in \mathrm{Out}(x) \quad \text{if and only if} \quad \phi(x) \in \mathrm{Out}(y). \tag{3.3}$$

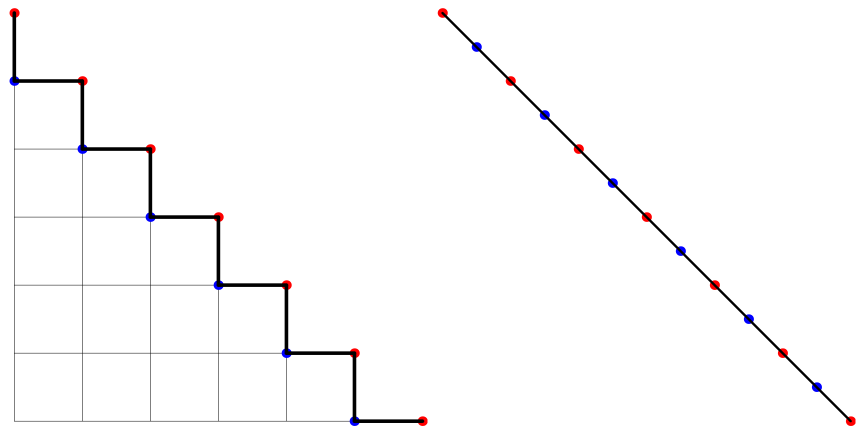

FIGURE 9. The doubling graph D, isomorphic to $\mathbb{Z}$, shown on the left in correspondence with two successive diagonals S_k, S_{k+1} of $\mathbb{Z}^2$.

Let D_k be the undirected bipartite graph with vertex set $S_k \cup S_{k+1}$, and an edge joining $x \in S_k$ and $y \in S_{k+1}$ if $y \in \mathrm{Out}(x)$.

The graphs D_k are isomorphic to each other for all $k \in \mathbb{Z}$. The **doubling graph** is a generic graph D that is isomorphic to each D_k. We can also interpret D as the image of $\mathbb{Z}^2$ under the equivalence relation $x \equiv \phi(x)$. In the case currently under discussion, we can take D to be simply $\mathbb{Z}$, with nearest-neighbour edges, as shown in Figure 9. Consider the map $v : \mathbb{Z}^2 \to \mathbb{Z}$ given by

$$v\big((x_1, x_2)\big) = x_1 - x_2. \tag{3.4}$$

Restricted to the set $S_k \cup S_{k+1}$, this gives an isomorphism between D_k and D, for any k.

Recall the definition of the hard-core model as given in Section 1.2; a Gibbs distribution for the hard-core model on a graph with vertex set W with activity $\lambda > 0$ is a distribution on configurations $\eta \in \{0,1\}^W$ satisfying (1.2).

Consider the hard-core model on the doubling graph D with vertex set $W = \mathbb{Z}$. This is a bipartite graph, with bipartition $W = W_0 \cup W_1$ where W_0 and W_1 are the sets of even and odd integers respectively. We consider the following two update procedures for configurations on $\{0,1\}^W$. For an "odd" update, for each vertex $x \in W_1$ independently, resample $\eta(x)$ according to the values at its two neighbours, setting $\eta(x) = 0$ with probability 1 if either of the neighbours takes value 1, and otherwise setting $\eta(x) = 1$ with probability $1 - p$. For an "even" update, do the same for vertices in W_0. Set $\lambda = 1/p - 1$, so that $1 - p = \lambda/(1 + \lambda)$. Since each of W_0 and W_1 is an independent set of D, any Gibbs distribution for the hard-core

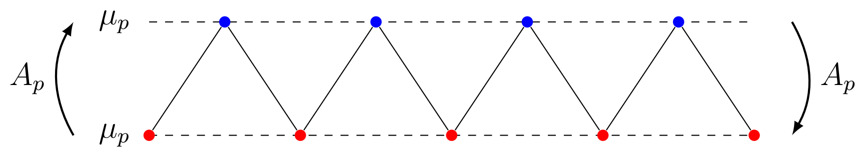

FIGURE 10. The Markovian distribution corresponding to a Gibbs measure for the hard-core model on the doubling graph W yields a Markovian distribution μ_p on each of the two vertex classes W_0 and W_1. Since the Gibbs distribution is invariant under the update procedures, the distribution μ_p is invariant for the PCA.

model with activity λ is invariant under both of these update operations. (This is a version of Glauber dynamics).

Take some even $k \in \mathbb{Z}$. Suppose we start from a configuration on $\{0,1\}^W$, which, via the isomorphism (3.4) between D and D_k under which W_0 maps to S_k and W_1 to S_{k+1}, corresponds to a configuration in $\{0,1\}^{S_k \cup S_{k+1}}$. Perform an odd update, resampling the sites of W_1, leading to a new configuration on $\{0,1\}^W$. Considering now (3.4) as an isomorphism between D_k and D_{k-1}, which maps W_0 to S_k and W_1 to S_{k-1}, the updated configuration on $\{0,1\}^W$ corresponds to a configuration in $\{0,1\}^{S_{k-1} \cup S_k}$, whose values at the sites in S_k are left unchanged. We can interpret the update as generating a configuration on S_{k-1} from a configuration on S_k. This procedure is identical to that which occurs in one iteration of the PCA A_p.

If we then perform an even update, resampling the sites of W_0, we can pass in the same way to a configuration on the sites of $S_{k-2} \cup S_{k-1}$, which corresponds to the next step of the PCA.

Continuing to perform odd and even updates alternately, we reproduce the evolution of the PCA. A Gibbs distribution on D is characterized by its marginal on the vertices of one half of the bipartition, say W_0. Since the distribution is preserved by the updates, this distribution on $\{0,1\}^{W_0}$ is 2-periodic for the PCA.

In fact, for any λ there is a unique Gibbs distribution for the hard-core model on $\mathbb{Z}$. Since the hard-core interaction is homogeneous and nearest-neighbour, this Gibbs distribution is itself a stationary Markov chain indexed by $\mathbb{Z}$. Let $Q = Q_p$ be its transition matrix. Therefore, the marginal distributions on W_0 and W_1 are in fact equal to each other. Call this marginal distribution μ_p. Then μ_p is the law of the stationary Markov chain with transition matrix $P = Q^2$. This μ_p is a stationary distribution for the PCA A_p, and the matrix P is the one in (3.1). See Figure 10 for an illustration.

As mentioned earlier, μ_p is a time-reversible stationary distribution for the evolution of the PCA. This follows from the reversibility of the process of configurations on G under the update procedure (this is essentially the standard reversibility property for Glauber dynamics and is easy to verify by checking the detailed balance equations). In fact, from the uniqueness of the Gibbs distribution for the hard-core model on G, one can deduce quite easily that there is only one reversible stationary distribution for the hard-core PCA A_p. However, this argument does not preclude the existence of other non-reversible stationary distributions. From Theorem 1, we know that such distributions do not in fact exist, but proving this required a different argument, as given in Section 2.4.

In contrast, in the next section we will use the implication in the other direction; in situations where there exist multiple Gibbs distributions for the hard-core model, we can conclude that there are multiple periodic distributions for the corresponding PCA; then the PCA is non-ergodic, and draws occur with positive probability in the corresponding game.

3.2. **General framework.** Recall that in the setting of Theorem 2, we have a locally finite graph G with vertex set V, along with a partition $(S_k : k \in \mathbb{Z})$ of V and an integer $m \geq 2$, such that conditions (A1) and (A2) given in Section 1.2 hold.

We also defined D_k be the graph with vertex set $S_k \cup \cdots \cup S_{k+m-1}$, with an undirected edge (x, y) whenever (x, y) is a (directed) edge of V. For convenience we will also use D_k to denote the vertex set $S_k \cup \cdots \cup S_{k+m-1}$.

Lemma 3.1. *The graphs D_k are isomorphic to each other for all $k \in \mathbb{Z}$.*

Proof. Consider the map χ_k defined on D_k under which

$$\chi_k(x) = \begin{cases} x & \text{if } x \in S_{k+1} \cup \cdots \cup S_{k+m-1} \\ \phi(x) & \text{if } x \in S_k \end{cases}.$$

From assumptions (A1) and (A2) above, χ_k is a graph isomorphism from D_k to D_{k+1}. Hence indeed D_k and D_{k+1} are isomorphic, and so by induction any two D_k, $D_{k'}$ are isomorphic. □

We then take D to be a graph isomorphic to any D_k. (When $m = 2$ we sometimes call D the **doubling graph**). Note that D is m-partite. Specifically, let us fix some isomorphism f_0 from D_0 to D, and let W_i be the image of S_i under f_0, for $i = 0, \ldots, m-1$. Then $(W_0, \ldots, W_{m-1})$ is a partition of the vertices of D into m classes, and assumption (A1) guarantees that there are no edges within a class W_i.

It will be important that we can map both D_k and D_{k+1} to D in such a way that the vertices common to D_k and D_{k+1} have the same image in both maps.

Lemma 3.2. *There exists a family of maps* $(f_k : k \in \mathbb{Z})$ *such that* f_k *is a graph isomorphism from* D_k *to* D*, and such that the following properties hold.*

(a) *For each* k*,*

$$f_k(x) = \begin{cases} f_{k+1}(x) & \text{for } x \in D_k \cap D_{k+1} = S_{k+1} \cup \cdots \cup S_{k+m-1} \\ f_{k+1}(\phi(x)) & \text{for } x \in S_k. \end{cases}$$

(b) *For each* k *and each* $r \in \{k, k+1, \dots, k+m-1\}$*, the image of* S_r *under* f_k *is* $W_{r \bmod m}$*.*

(c) *Let* $x \in S_k$ *and* $y \in D_k = S_k \cup \cdots \cup S_{k+m-1}$*. Then* $y \in \mathrm{Out}(x)$ *if and only if* $f_k(y)$ *is a neighbour of* $f_k(x)$ *in* D*.*

Proof. Let f_0 be the isomorphism from D_0 to D described just above. Then we can compose f with the isomorphisms χ_k defined in the proof of Lemma 3.1, by setting

$$f_k = \begin{cases} f_0 \circ \chi_{-1} \circ \chi_{-2} \circ \cdots \circ \chi_k & \text{for } k < 0 \\ f_0 \circ \chi_0^{-1} \circ \chi_1^{-1} \circ \cdots \circ \chi_{k-1}^{-1} & \text{for } k > 0. \end{cases}$$

Then using assumption (A2), it is easy to check by induction upwards and downwards from 0 that f_k is an isomorphism from D_k to D satisfying the properties stated in (a) and (b), for each k.

Finally note that by (A1), if $x \in S_k$ then $\mathrm{Out}(x) \subseteq S_k \cup \cdots \cup S_{k+m-1}$ while $\mathrm{In}(x)$ is disjoint from $S_k \cup \cdots \cup S_{k+m-1}$. By definition, the set of neighbours of x in the graph D_k is then $\mathrm{Out}(x)$. Then part (c) follows since f_k is a graph isomorphism from D_k to D. □

Given a hard-core configuration in $\{0,1\}^D$, we can consider Glauber update steps that resample the vertices of one of the vertex classes $W_0, W_1, \dots, W_{m-1}$. To perform an update of the class W_i: for each $v \in W_i$ independently, let the new value at v be 0 if any neighbour of x has value 1, and otherwise let the new value at v be 0 with probability $p = 1/(1+\lambda)$ and 1 with probability $1-p = \lambda/(1+\lambda)$. If a distribution on $\{0,1\}^D$ is a Gibbs distribution for the hard-core model on D with activity $\lambda = 1/p - 1$, then it is invariant under this update procedure for each $i = 0, 1, \dots, m-1$. (Again, this is a version of the Glauber dynamics for the hard-core model on D.)

Proof of Theorem 2. We start by defining an analogue of the hard-core PCA A_p in the general setting. As before, we have the recursion (2.1) for the outcome of the game started from $x \in V$, in terms of the outcomes started from the elements of $\mathrm{Out}(x)$ together with the information about whether x itself is open or closed. (Recall that we treat the game from x as a win if x is closed.)

As in previous sections we can specialize that recursion to configurations involving only the symbols $0 = W$ and $1 = L$. This gives the following recursion for a family of variables $(\gamma(x) : x \in V) \in \{0,1\}^V$ (which we do not assume to be necessarily game outcomes):

$$x \text{ closed} \;\Rightarrow\; \gamma(x) = 0;$$

$$x \text{ open} \;\Rightarrow\; \gamma(x) = \begin{cases} 1 & \text{if } \gamma(y) = 0 \text{ for all } y \in \text{Out}(x) \\ 0 & \text{if } \gamma(y) = 1 \text{ for some } y \in \text{Out}(x). \end{cases} \tag{3.5}$$

If $x \in S_k$, then $\text{Out}(x) \subset S_{k+1} \cup \dots \cup S_{k+m-1}$. Thus, the recursion (3.5) gives $(\gamma(x) : x \in S_k)$ in terms of $(\gamma(x) : x \in S_{k+1} \cup \dots \cup S_{k+m-1})$ and the random configuration of closed and open sites in S_k (which we take as usual to be product measure with each site closed with probability p). This is analogous to the PCA A_p considered earlier (although for $m > 2$, a "state" of the PCA is now more complicated to describe).

Fix $K \in \mathbb{Z}$, and take some boundary condition $(\gamma(x) : x \in S_K \cup \dots \cup S_{K+m-1})$, which we allow to be be random, but which is independent of the configuration of open and closed sites in $\bigcup_{r<K} S_r$. Applying (3.5) repeatedly then generates an evolution $(\gamma(x) : x \in S_r)_{r \le K+m-1}$.

We will couple this evolution with a process of configurations in $\{0,1\}^D$. For $k \le K$ and $v \in D$, define $\sigma_k(v) = \gamma(f_k^{-1}(v))$. Then $\sigma_k \in \{0,1\}^D$ for each k. The idea is now to show that the transformation from σ_{k+1} to σ_k is identical to a hard-core update of the vertex class $W_{k \bmod m}$, with randomness provided by the configuration of open and closed vertices in S_k. Notice that σ_{k+1} is a function of $(\gamma(x) : x \in S_{k+1} \cup \dots \cup S_{k+m})$, while σ_k is a function of $(\gamma(x) : x \in S_k \cup \dots \cup S_{k+m-1})$. If $v \in W_i$ where $i \ne k \bmod m$, then by Lemma 3.2(a) and (b), $f_{k+1}^{-1}(v) = f_k^{-1}(v) \in S_{k+1} \cup \dots \cup S_{k+m-1}$. Thus $\sigma_{k+1}(v) = \sigma_k(v)$. So the only sites in D which can change their value between the configuration σ_{k+1} and the configuration σ_k are those in $W_{k \bmod m}$. Consider such a $v \in W_{k \bmod m}$, and let $x = f_k^{-1}(v)$ so that $x \in S_k$ (by Lemma 3.2(b)) and $\sigma_k(v) = \gamma(x)$.

Translating (3.5) and using Lemma 3.2(c) gives that for $v \in W_{k \bmod m}$:

$$x \text{ closed} \;\Rightarrow\; \sigma_k(v) = 0;$$

$$x \text{ open} \;\Rightarrow\; \sigma_k(v) = \begin{cases} 1 & \text{if } \sigma_{k+1}(u) = 0 \text{ for all } u \sim v \\ 0 & \text{if } \sigma_{k+1}(u) = 1 \text{ for some } u \sim v. \end{cases} \tag{3.6}$$

Each bit of randomness (the information about whether $x \in S_k$ is open or closed) is used only once. Since each x is closed with probability p independently, we have that the conditional distribution of σ_k given $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_K$ is precisely that obtained by performing a hard-core update of the vertex class $W_{k \bmod m}$.

Now let μ be a Gibbs distribution for the hard-core model on D. By choosing the distribution of the boundary condition $(\gamma(x) : x \in S_K \cup \cdots \cup S_{K+m-1})$ correspondingly, we can arrange that σ_K has distribution μ. But then since μ is invariant under the hard-core updates, σ_k has distribution μ for all $k < K$ also.

So, suppose that there exist multiple Gibbs distributions, and let μ and ν be two of them. By alternating between μ and ν, we can arrange a sequence indexed by K of boundary conditions $(\gamma^{(K)}(x) : x \in S_K \cup \cdots \cup S_{K+m-1})$ that induces a sequence of distributions of $\sigma_0^{(K)}$ having both μ and ν as limit points as $K \to \infty$. In particular, the configuration $\sigma_0^{(K)}$ does not converge almost surely as $K \to \infty$ (in the product topology). So the sequence of configurations $(\gamma^{(K)}(x) : x \in S_0 \cup \cdots \cup S_{m-1})$ does not converge almost surely.

Now we apply the same argument that we used for the second part of the proof of Proposition 2.2. If the game started from x is not a draw, then one player has a strategy which guarantees a win in fewer than N moves, where $N \in \mathbb{N}$ is an almost surely finite random variable which depends on the percolation configuration. Then the values $\gamma^{(K)}(x)$ must agree for all large enough K. Hence if there is zero probability of a draw from each site, then the configuration $(\gamma^{(K)}(x) : x \in S_0 \cup \cdots \cup S_{m-1})$ converges almost surely as $K \to \infty$. By the argument in the previous paragraph, this contradicts the existence of multiple hard-core Gibbs distributions. □

3.3. **Remarks on the converse direction.** We do not know whether the converse of Theorem 2 holds; that is, whether the uniqueness of the hard-core Gibbs distribution on D implies that there are no draws on G. Two separate issues arise.

First, assume that the hard-core Gibbs distribution is unique, and suppose we wish to conclude that there is a unique distribution of the evolution $(\gamma(x) : x \in G) \in \{0,1\}^V$ satisfying (3.5) that is stationary in the sense that $(\gamma(x) : x \in G)$ and $(\gamma(\phi(x) : x \in G)$ have the same law. Uniqueness of the hard-core distribution gives only that there is a unique such law satisfying an additional condition, which is that σ_k has the same distribution for all k (and not merely for any two k which agree $\bmod m$). In the case $m = 2$, this condition can be interpreted as a reversibility property, although the precise meaning of this is complicated by the lack of a canonical bijection between S_k and S_{k+1}. Specifically, the condition is equivalent to the statement that the ordered pair $[(\gamma(x) : x \in S_0), (\gamma(y) : y \in S_{-1})]$ has the same distribution as $[(\gamma(x) : x \in S_0), (\gamma\phi(y) : y \in S_{-1})]$. In the example of the game in $\mathbb{Z}^2$, this is also equivalent to symmetry of the law of the evolution on $\mathbb{Z}^2$ under reflection in the line $x_1 + x_2 = 0$.

However, there may be other stationary evolutions that do not satisfy such a reversibility condition. (As mentioned before, our argument for $\mathbb{Z}^2$ rules these out by a different method.)

Second, note that when G is bipartite (for example, whenever $m = 2$), the uniqueness of law of a stationary evolution with the binary alphabet $\{0, 1\}$ is enough to imply that the probability of draws is zero. This follows by an argument similar to the proof of Proposition 2.1, showing that the PCA on three symbols $\{0, ?, 1\}$ is ergodic whenever the binary PCA is ergodic.

However, if G is not bipartite (so that there are sites which can be reached on the turn of either one of the players), then this argument breaks down. The monotonicity associated to the stochastic order generated by $0 \lhd ? \rhd 1$ (see Lemma 2.1(ii)) still applies, but the monotonicity associated to the stochastic order generated by $0 < ? < 1$ (see Lemma 2.1(i)) fails; the operation analogous to a single application of the operator F_p involves moving from a configuration on the vertices $S_{k+1} \cup \cdots \cup S_{k+m}$ to a configuration on the vertices $S_k \cup \cdots \cup S_{k+m-1}$, and these two sets overlap when $m > 2$.

Hence it is no longer clear that uniqueness for the binary PCA would imply the same for the three-symbol PCA, as required to conclude that there are no draws. Closely related examples in which ergodicity of a binary PCA does not imply ergodicity of its envelope PCA are noted in [BMM13].

3.4. **Example graphs with $d \geq 3$.** We now give several examples of graphs G to which one may hope to apply Theorem 2. (As we will see, the result can indeed be applied in some cases, but not in others.) We consider the vertex set $V = \mathbb{Z}^d$ (or subsets thereof), for various different choices of the set $\mathrm{Out}(x)$ of vertices to which the token can move from x. In each case we consider the percolation game in which each site is closed with probability p and open with probability $1 - p$. All our examples can be regarded as natural extensions of the original $\mathbb{Z}^2$ game, in the sense that they reduce to it when we set $d = 2$.

Example 3.1. Let $\mathrm{Out}(x) = \{x + e_i : 1 \leq i \leq d\}$. So $|\mathrm{Out}(x)| = d$. This is perhaps the most natural extension of all. However we cannot apply Theorem 2 because there is no choice of the automorphism ϕ for which assumption (A2) holds.

Example 3.2. Let $\mathrm{Out}(x) = \{x \pm e_i + e_d : 1 \leq i \leq d - 1\}$. This is the example already mentioned in Section 1.2. Here $|\mathrm{Out}(x)| = 2(d - 1)$. Since any step preserves parity, it is natural to restrict to the set of even sites $\mathbb{Z}^d_{\text{even}} := \{x \in \mathbb{Z}^d : \sum x_i \text{ is even}\}$.

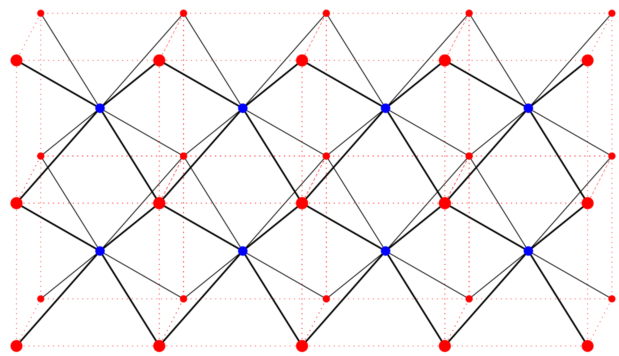

FIGURE 11. The three-dimensional body-centred cubic lattice (which is the doubling graph for the PCA associated to Example 3.3 when $d = 4$). The two underlying copies of $\mathbb{Z}^3$ are shown with red and with blue vertices. The black lines are the edges of the body-centred cubic lattice, and the dotted red lines show the nearest-neighbour edges in the red copy of $\mathbb{Z}^3$.

In two dimensions, the game is isomorphic to the original game on $\mathbb{Z}^2$. For general d, conditions (A1) and (A2) hold with $m = 2$ if we set $S_k = \{x \in \mathbb{Z}^d_{\text{even}} : x_d = k\}$ and $\phi(x) = x + 2e_d$.

To obtain the doubling graph, consider $D_k = S_k \cup S_{k+1}$ with an edge between $x \in S_k$ and $y \in S_{k+1}$ whenever $y \in \text{Out}(x)$. This gives a graph isomorphic to the standard cubic lattice $\mathbb{Z}^{d-1}$ (for example, the map

$$(x_1, \dots, x_{d-1}, x_d) \to (x_1, \dots, x_{d-1})$$

gives a graph isomorphism from $D_0 = S_0 \cup S_1$ to $\mathbb{Z}^{d-1}$).

The graph is vertex-transitive. By Theorem 2, if there exist multiple Gibbs distributions for the hard-core model on $\mathbb{Z}^{d-1}$ with activity λ, then the percolation game on G with $p = 1/(1+\lambda)$ has positive probability of a draw from any vertex.

Example 3.3. Now let $\text{Out}(x) = \{x \pm e_1 \pm e_2 \cdots \pm e_{d-1} + e_d\}$, so that $|\text{Out}(x)| = 2^{d-1}$. Each step changes the parity of every coordinate, so we restrict to the set $\mathbb{Z}^d_{\text{bcc}} = \{x \in \mathbb{Z}^d : x_i \equiv x_j \bmod 2 \text{ for all } i, j\}$. Putting an edge between x and y whenever $y \in \text{Out}(x)$, we obtain the **body-centred cubic lattice** in d dimensions. This consists of two copies of $(2\mathbb{Z})^d$, each offset from the other by $(1, 1, \dots, 1)$, so that each point of one lies at the centre of a unit cube of the other; the edges are given by joining each point to the 2^d corners of the surrounding unit cube. See Figure 11 for an illustration.

Conditions (A1) and (A2) hold for $m = 2$ with $S_k = \{x \in \mathbb{Z}^d_{\text{bcc}} : x_d = k\}$ and $\phi(x) = x + 2e_d$. The doubling graph D isomorphic to $D_k = S_k \cup S_{k+1}$ for

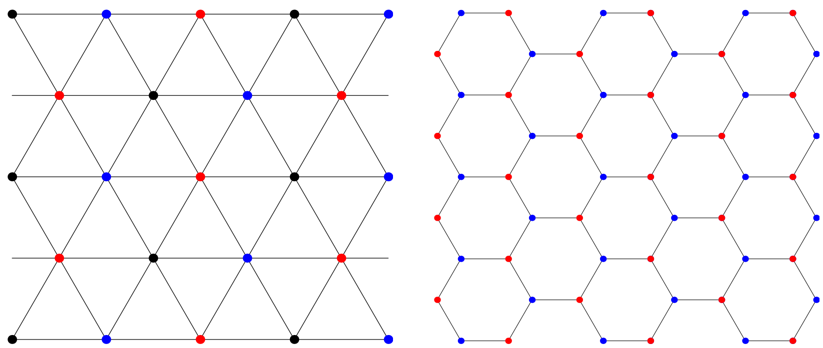

FIGURE 12. Triangular lattice and hexagonal lattice.

each k is now the body-centered cubic lattice in $d-1$ dimensions. The map $v(x) = (x_1, x_2, \dots, x_{d-1})$ from $\mathbb{Z}^d_{\mathrm{bcc}}$ to $\mathbb{Z}^{d-1}_{\mathrm{bcc}}$ restricts to an isomorphism between D_k and D for each k.

When $d=2$ or $d=3$ the graph G is isomorphic to that in Example 3.2 above, but for $d \geq 4$ the graphs are different. Existence of multiple hard-core distributions on $\mathbb{Z}^{d-1}_{\mathrm{bcc}}$ will imply existence of draws on $\mathbb{Z}^d_{\mathrm{bcc}}$.

Example 3.4. Let $\operatorname{Out}(x) = \{x + \sum_{i\in S} e_i : S \subset \{1,\dots,d\} \text{ with } 1 \leq |S| \leq d-1\}$. So a move of the game corresponds to incrementing at least one, but not all, of the coordinates by one. Then $|\operatorname{Out}(x)| = 2^d - 2$.

Conditions (A1) and (A2) hold with $m = d$ if we set $S_k = \{x : \sum x_i = k\}$ and $\phi(x) = x + e_1 + e_2 + \cdots + e_d$. For $d = 2$ the game is the same as ever. For $d > 2$ there are some new features. For the first time we have $m > 2$, and the graph G is not bipartite; from a given starting vertex, there are vertices that can be reached when it is either player's turn. The graph D is $(d-1)$-dimensional and d-partite. For $d = 3$, it corresponds to the triangular lattice. For example, the map

$$(x_1, x_2, x_3) \to x_1(1,0) + x_2\left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right) + x_3\left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right) \tag{3.7}$$

is an isomorphism from $D_k = S_k \cup \cdots \cup S_{k+m-1}$ to the triangular lattice for each k.

Example 3.5. Fix r with $1 \leq r \leq d$, and now restrict to sites $x \in \mathbb{Z}^d$ such that $\sum x_i \equiv 0$ or $r \bmod d$. For x with $\sum x_i \equiv 0 \bmod d$, let $\operatorname{Out}(x) = \{x + \sum_{i\in S} e_i,$ for any $S \subset \{1,\dots,d\}$ with $|S| = r\}$. Meanwhile, for x with $\sum x_i \equiv r \bmod d$, let $\operatorname{Out}(x) = \{x + \sum_{i\in S} e_i,$ for any $S \subset \{1,\dots,d\}$ with $|S| = d-r\}$.

Now $|\operatorname{Out}(x)| = \binom{d}{r}$ for all x. Replacing r by $d-r$ gives an isomorphic graph, so we may assume $1 \leq r \leq d/2$. Then conditions (A1) and (A2) hold with $m = 2$,

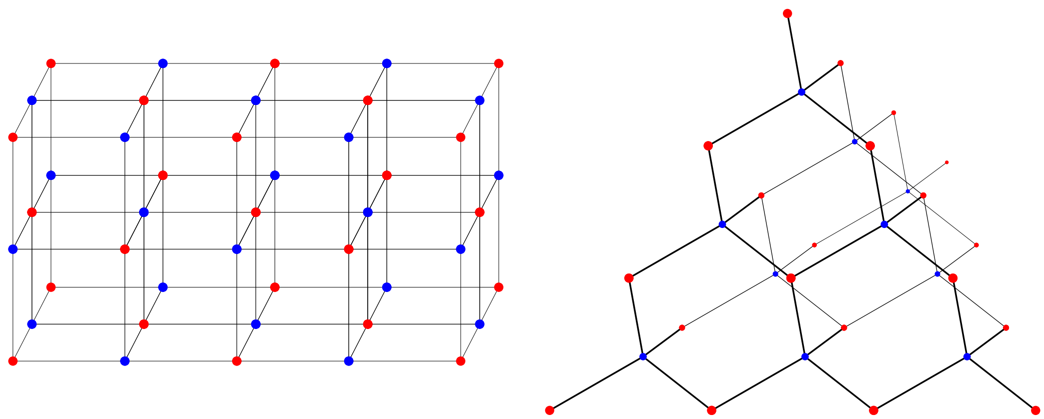

FIGURE 13. The cubic lattice and the diamond cubic graph.

with $\phi(x) = x + \sum_{i=1}^{d} e_i$, and with $S_k = \{x : \sum x_i = dk/2\}$ for even k and $S_k = \{x : \sum x_i = d(k-1)/2 + r\}$ for odd k.

For $d = 2$ (and hence $r = 1$) the game is the familiar two-dimensional game. For $d = 3$ and $r = 1$, we get $|\operatorname{Out}(x)| = 3$ and the doubling graph D is the two-dimensional hexagonal lattice; this is the image of $\{x \in \mathbb{Z}^d : \sum x_i \equiv 0 \text{ or } 1 \bmod 3\}$, with edges between x and y where $y \in \operatorname{Out}(x)$, under the map (3.7) above.

For $d = 4$ and $r = 2$, the graph G is isomorphic to the $d = 4$ case of Example 3.2 above, and so D is the standard cubic lattice $\mathbb{Z}^3$. For $d = 4$ and $r = 1$, we have $|\operatorname{Out}(x)| = 4$, and D is the so-called **diamond cubic graph** (see for example Section 6.4 of [CS99]). This graph may, for example, be represented as

$$\left\{y \in \mathbb{Z}^3 : y_1 \equiv y_2 \equiv y_3 \bmod 2 \text{ and } y_1 + y_2 + y_3 \equiv 0 \text{ or } 1 \bmod 4\right\},$$

with edges between nearest neighbours (which are at distance $\sqrt{3}/4$). This is the image of $\{x \in \mathbb{Z}^d : \sum x_i \equiv 0 \text{ or } 1 \bmod 3\}$, with edges between x and y where $y \in \operatorname{Out}(x)$, under the map

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 - x_3 + x_4 \\ -x_1 + x_2 - x_3 + x_4 \\ x_1 + x_2 - x_3 - x_4 \end{pmatrix}$$

(see [NS08]). See Figure 13 for an illustration.

Theorem 3. *There is positive probability of a draw from every vertex for sufficiently small p in the following cases: Example 3.2 for all $d \geq 3$; Example 3.3 for $d = 3$ and $d = 4$; Example 3.4 for $d = 3$; and Example 3.5 for $d = 3$ (with $r = 1$) and $d = 4$ (with $r = 1$ or $r = 2$).*

Proof. In the cases listed, it is known that there exist multiple Gibbs distributions for the hard-core model on the associated graph D when the activity parameter λ is sufficiently high. For the standard cubic lattice in any dimension greater than 1, the result goes back to Dobrushin [Dob65]. Other models in two and three dimensions were covered by Heilmann [Hei74] and Runnels [Run75], including the triangular and hexagonal lattices in two dimensions and the body-centered cubic lattice and the diamond cubic graph in three dimensions.

Theorem 2 shows that there is positive probability of a draw for small enough p from some vertex, and since all the graphs G are vertex-transitive, the conclusion holds for every vertex. □

It is expected that in fact the hard-core model on D has multiple Gibbs distributions for λ sufficiently large in all of Examples 3.2–3.5 whenever $d \geq 3$ (so that D has dimension at least 2). This could likely be proved by Peierls contour arguments, although this requires a suitable definition of a contour, which is typically graph-dependent, and less straightforward than in other settings such as the Ising model. Via Theorem 2, such non-uniqueness would imply existence of draws for the corresponding graphs G.

We emphasize again that there is a more fundamental obstacle to proving existence of draws for the standard oriented lattice $\mathbb{Z}^d$ of Example 3.1, in that our condition (A2) does not hold here.

3.5. **Extending the hard-core correspondence.** Various further extensions can be made while still preserving the correspondence to the hard-core model. For example, in the class of models considered in Theorem 2, we can augment the set of allowable moves from site x to include the point $\phi(x)$ itself.

Specifically, replace (A1) and (A2) by the following assumptions:

- (A1′) For all $x \in S_k$, $\mathrm{Out}(x) \subset S_{k+1} \cup \cdots \cup S_{k+m-1} \cup \{\phi(x)\}$.
- (A2′) There is a graph automorphism ϕ of G that maps S_k to S_{k+m} for every k, such that $\phi(x) \in \mathrm{Out}(x)$ for all x, and such that $\mathrm{Out}(x) \setminus \{\phi(x)\} = \mathrm{In}(\phi(x)) \setminus x$.

Define D as before; D is a graph isomorphic to any D_k, where D_k is the graph with vertex set $S_k \cup \cdots \cup S_{k+m-1}$ and an undirected edge (x, y) whenever (x, y) is a directed edge of V. Note now an edge $(x, \phi(x))$ in G does not give a corresponding edge in D.

Proposition 3.1. *Suppose that the graph G satisfies (A1′) and (A2′). If there are multiple Gibbs distributions for the hard-core model on D with activity $\lambda < 1$. then the percolation game on G with $p = 1 - \lambda$ has positive probability of a draw from some vertex.*

The method of proof is a slight variation on that of Theorem 2, which we indicate briefly. To reflect the presence of the edge $(x, \phi(x))$, we change the hard-core update procedure. When we perform an update of the vertex class W_i, we now add that any vertex $v \in W_i$ which is in state 1 before the update must move to state 0 after the update; otherwise the update at v proceeds as before.

Again one can show that hard-core Gibbs distributions are stationarity under such updates, but now with the activity parameter λ equal to $1-p$ rather than $1/p-1$ as previously. (To verify the stationarity, one can start by checking the detailed balance condition for an update at a single site; then if the distribution is stationary for the update at any single site, it is also invariant under simultaneous updates at any set of non-neighbouring sites.)

Note that now when $p \to 0$, we have $\lambda \to 1$ rather than $\lambda \to \infty$. Hence to show existence of draws for some p, we need multiplicity of Gibbs distributions for some $\lambda < 1$. For the case of the standard cubic lattice, Galvin and Kahn [GK04] show that this holds for sufficiently high dimension, so that we can deduce the existence of draws for the variant of Example 3.2 in which $\mathrm{Out}(x) = \{x \pm e_i + e_d : 1 \le i \le d-1\} \cup \{x + e_d\}$, when d is sufficiently large.

4. Open questions

4.1. **The oriented cubic lattice.** For the percolation game on $\mathbb{Z}^d$ (where the allowed moves from site x are to any open site $x + e_i$ for $1 \le i \le d$), do there exist any $p \in (0,1)$ and $d \ge 3$ for which draws occur with positive probability?

4.2. **Monotonicity and phase transition.** For lattices where draws are known to occur (for example, the even sublattice of $\mathbb{Z}^d$ with $d \ge 3$ and moves allowed from x to any open $x + e_d \pm e_i$ for $1 \le i \le d-1$), is the probability of a draw starting from the origin non-increasing in the density p of closed sites? Or, at least, is the set of p that have positive draw probability a single interval containing 0 (so that there is a single critical point at the upper end of the interval)? If so, what happens at the critical point? (It is also unknown whether the hard core model on $\mathbb{Z}^d$ has a single critical point for uniqueness of Gibbs distributions; see e.g. [BGRT13] for discussion and recent bounds.)

4.3. **Reversibility.** For lattices where draws are known to occur, such as the example described in question 4.2 above, do there exist $\{0,1\}$-valued invariant distributions for the game evolution that are not projections of a hard-core Gibbs distribution on the graph D, and therefore lack the corresponding reversibility property? Or is it the case that the game evolution has multiple invariant distributions precisely when there are multiple hard-core Gibbs distributions?

4.4. **Target game win probability.** Compute the winning probability for the target game on $\mathbb{Z}^2$ (in which the winner is the first player to move to a closed site).

4.5. **Misère games.** Besides our target game, there are other natural misère variants. For example, suppose as before that moves are allowed from $x \in \mathbb{Z}^2$ to an open site in $\{x + e_1, x + e_2\}$, but now declare that if both these sites are closed, then the player whose turn it is to move *wins*. Does this game have zero probability of a draw for all p? Unlike the original percolation game or the target game, there is apparently no useful correspondence to a PCA with alphabet $\{0, 1\}$.

4.6. **Elementary probabilistic cellular automata.** Is every elementary PCA (i.e. one with 2 states and a size-2 neighhborhood) on $\mathbb{Z}$ with positive rates ergodic? Can our weighting approach be extended to prove ergodicity for other PCA in this class?

4.7. **Undirected lattices.** The following game is considered in [BHMW15]. As usual, each site of $\mathbb{Z}^d$ is independently closed with probability p, and two players alternately move a token. From an open site x, a move is permitted to *any* open nearest neighbour $x \pm e_i$ *provided it has not been visited previously.* A player who cannot move loses. This game is closely related to maximum matchings, and this is used in [BHMW15] to derive results for biased variants in which odd and even sites have different percolation parameters. However, for the unbiased version described above it is unknown whether there exist any $p > 0$ and $d \geq 2$ for which draws occur with positive probability.

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