EXTENDABLE SELF-AVOIDING WALKS

GEOFFREY R. GRIMMETT, ALEXANDER E. HOLROYD,
AND YUVAL PERES

Abstract. The connective constant μ of a graph is the exponential growth rate of the number of n-step self-avoiding walks starting at a given vertex. A self-avoiding walk is said to be *forward* (respectively, *backward*) *extendable* if it may be extended forwards (respectively, backwards) to a singly infinite self-avoiding walk. It is called *doubly extendable* if it may be extended in both directions simultaneously to a doubly infinite self-avoiding walk. We prove that the connective constants for forward, backward, and doubly extendable self-avoiding walks, denoted respectively by μ^{F}, μ^{B}, μ^{FB}, exist and satisfy $\mu = \mu^{\mathrm{F}} = \mu^{\mathrm{B}} = \mu^{\mathrm{FB}}$ for every infinite, locally finite, strongly connected, quasi-transitive directed graph. The proofs rely on a 1967 result of Furstenberg on dimension, and involve two different arguments depending on whether or not the graph is unimodular.

1. Introduction

Let $G = (V, E)$ be an infinite, strongly connected, locally finite, directed graph (possibly with parallel edges), and let $\sigma_n(v)$ be the number of n-step self-avoiding walks (SAWs) on G starting at the vertex $v \in V$ and directed away from v. Hammersley proved in 1957 [5] that the limit

$$\mu := \lim_{n\to\infty} \Big(\sup_{v\in V} \sigma_n(v)\Big)^{1/n} \tag{1.1}$$

exists, and that if G is quasi-transitive then

$$\lim_{n\to\infty} \sigma_n(v)^{1/n} = \mu \qquad \text{for all } v \in V. \tag{1.2}$$

The constant $\mu = \mu(G)$ is called the **connective constant** of G. Note that (1.1) is not necessarily the natural definition of connective constant for a general graph, see [4, 8]. There is no loss of generality in restricting attention to directed graphs, since each edge of an

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undirected graph may be interpreted as a pair of edges with opposite orientations.

The purpose of this article is to study the growth rates of the numbers of n-step SAWs from v that are extendable to infinite SAWs at one or both of their ends.

Let w be an n-step directed SAW starting at a vertex v. We call w **forward extendable** if it is an initial segment of some singly infinite directed SAW from v. We call w **doubly extendable** if it is a sub-walk of some doubly infinite directed SAW passing through v. We call w **backward extendable** if it is the final segment of some singly infinite directed SAW from infinity, passing through v and ending at the other endpoint of w. Let $\sigma_n^{\mathrm{F}}(v)$, $\sigma_n^{\mathrm{B}}(v)$, and $\sigma_n^{\mathrm{FB}}(v)$ denote the numbers of forward, backward, and doubly extendable n-step SAWs from v, respectively. We define μ^{F}, μ^{B}, and μ^{FB} analogously to (1.1) whenever the limits exist.

Theorem 1. *Let G be an infinite, locally finite, strongly connected, quasi-transitive directed graph.*

(i) *The limits μ^{F}, μ^{B}, μ^{FB} exist and satisfy $\mu = \mu^{\mathrm{F}} = \mu^{\mathrm{B}} = \mu^{\mathrm{FB}}$.*

(ii) *We have*

$$\lim_{n\to\infty} \sigma_n^{\mathrm{F}}(v)^{1/n} = \mu^{\mathrm{F}} \qquad \text{for all } v \in V. \tag{1.3}$$

The analogue of (1.3) does not hold in general for backward or doubly extendable walks. For example, if v has only one neighbour (joined to it by edges in both directions) then $\sigma_n^{\mathrm{B}}(v)$ and $\sigma_n^{\mathrm{FB}}(v)$ are both 0 for $n \geq 1$.

A principal ingredient of the proof of Theorem 1 is a result of Furstenberg [3, Prop. III.1] from 1967; a recent exposition appears in [10, Sect. 3.3]. The same method provides an alternative proof of Hammersley's result (1.2).

Theorem 1(i) states that the exponential growth rates of counts of SAWs coincide for the four types of SAW under consideration. One may ask also about more refined asymptotic properties. Suppose G satisfies the conditions of Theorem 1, and is for simplicity vertex-transitive. The sub-multiplicativity of SAW-counts (see the proof of Lemma 5 below) gives that $\sigma_n^{\bullet} \geq (\mu^{\bullet})^n$ for each of $\bullet \in \{\ \ , \mathrm{F}, \mathrm{B}, \mathrm{FB}\}$. Therefore, whenever it is known that $\sigma_n \leq A\mu^n$ for some $A < \infty$, it follows that $A^{-1} \leq \sigma_n^{\bullet}/\sigma_n \leq 1$. This is indeed the situation for the (undirected) integer lattice $\mathbb{Z}^d$ with $d \geq 5$, by [7, Thm 1.1(a)]. We do not know whether $\sigma_n^{\bullet}$ and σ_n agree up to a multiplicative constant for every G satisfying the conditions of Theorem 1. The square lattice $\mathbb{Z}^2$ is a particularly interesting case.

In the case $G = \mathbb{Z}^d$ with $d \geq 2$, the method of 'bridges' developed by Hammersley and Welsh [6] immediately gives the results of Theorem 1, and furthermore shows that $\sigma_n^\bullet/\sigma_n \geq \exp(-c\sqrt{n})$ for some $c = c(d) > 0$. An interesting related notion of 'endless SAWs' is studied in [2].

The proof of Theorem 1 is divided into several parts. The proofs of the equalities $\mu = \mu^{\mathrm{F}}$ and $\mu^{\mathrm{B}} = \mu^{\mathrm{FB}}$ of part (i) use a result of Furstenberg, namely that a subperiodic tree has growth rate equal to its branching number. This is applied to certain trees constructed from the sets of SAWs (of the various types) from a given vertex, and it is argued that the branching numbers of the appropriate pairs of trees coincide. See Section 3. A related argument gives part (ii). The remaining equality is proved by two different arguments depending on whether or not G is unimodular. In the unimodular case, a mass-transport argument yields $\mu^{\mathrm{F}} = \mu^{\mathrm{B}}$ (Section 4), while in the non-unimodular case (Section 5) we show the existence of a 'quasi-geodesic' of a certain type, and employ a counting argument related to Hammersley's methods of [5] to obtain $\mu = \mu^{\mathrm{B}}$. In Section 2 below we define the various concepts referred to above. Many of our arguments can be simplified if G is undirected and/or transitive, and we indicate such simplifications where appropriate.

2. Preliminaries

In this section we introduce terminology and our main tools. Let $G = (V, E)$ be a directed graph, possibly with parallel edges. Let $\mathrm{Aut}(G)$ denote its automorphism group (where automorphisms are required to preserve edge orientations). The orbits of V under $\mathrm{Aut}(G)$ are called **transitivity classes**, and G is **transitive** if it has only one transitivity class, or **quasi-transitive** if it has finitely many transitivity classes. We call G **locally finite** if each vertex has finite in-degree and out-degree.

A **walk** w consists of a sequence of vertices $(v_i)_{m<i<n}$ together with edges $(e_i)_{m<i<i+1<n}$, where e_i is a directed edge from v_i to v_{i+1}, and where $-\infty \leq m \leq n \leq \infty$. The length $|w|$ of w is the number of its edges. The walk is **singly infinite** if either $m \in \mathbb{Z}$ and $n = \infty$ or $m = -\infty$ and $n \in \mathbb{Z}$, and **doubly infinite** if $m = -\infty$ and $n = \infty$. A graph G is **strongly connected** if for every pair $u, v \in V$ there exist finite walks from u to v and from v to u.

A **self-avoiding walk** (SAW) on G is a walk all of whose vertices are distinct. SAWs may be finite, singly infinite, or doubly infinite. Let $\sigma_n(v) = \sigma_n(v, G)$ be the number of length-n SAWs starting at $v \in V$. In the presence of parallel edges, two SAWs with identical vertex-sequences but different edge-sequences are considered distinct. We write $\sigma_n = \sigma_n(G) := \sup_{v\in V} \sigma_n(v)$, and denote

$$\mu = \mu(G) := \lim_{n\to\infty} \sigma_n^{1/n},$$

whenever the limit exists. Forward, backward, and doubly extendable SAWs are defined as in the introduction, and the quantities $\sigma_n^{\mathrm{F}}(v)$, σ_n^{F}, μ^{F}, etc., are defined analogously.

We turn now to certain elements in the study of trees, for which we follow [10, Chap. 3]. Let $T = (W, F)$ be an infinite, locally finite tree with root o. For $v \in W$, the distance between v and o is written $|v|$. For $e = \langle v_1, v_2\rangle \in F$, let $|e| = \max\{|v_1|, |v_2|\}$. Let $W_n = \{v \in W : |v| = n\}$ be the set of vertices at **level** n. A **cutset** is a minimal set of edges whose removal leaves o in a finite component. Since T is assumed locally finite, cutsets are finite.

There are two natural notions of dimension of a tree T. The **growth** is given by

$$\mathrm{gr}(T) := \lim_{n\to\infty} |W_n|^{1/n},$$

whenever this limit exists. In any case, the lower growth and upper growth are given respectively by

$$\underline{\mathrm{gr}}(T) := \liminf_{n\to\infty} |W_n|^{1/n}, \qquad \overline{\mathrm{gr}}(T) := \limsup_{n\to\infty} |W_n|^{1/n}.$$

A more refined notion is the **branching number**

$$\mathrm{br}(T) := \sup\Big\{\lambda : \inf_{\Pi} \sum_{e\in\Pi} \lambda^{-|e|} > 0\Big\}, \tag{2.1}$$

where the infimum is over all cutsets Π of T. One interesting property, which is sometimes helpful for intuition, is that the critical probability $p_{\mathrm{c}}(T)$ of bond percolation on T satisfies

$$p_{\mathrm{c}}(T) = 1/\mathrm{br}(T). \tag{2.2}$$

See [9, Thm 6.2] or [10, Thm 5.15].

The growth and branching number of a general tree need not be equal (and indeed the growth need not exist). However, we have the following inequality [10, eqn (1.1)]. We include a proof for the reader's convenience.

Lemma 2. *For any locally finite, infinite rooted tree,* $\mathrm{br}(T) \le \underline{\mathrm{gr}}(T)$.

Proof. Let $\lambda > \underline{\mathrm{gr}}(T)$. Taking Π as the set of edges joining W_{n-1} and W_n, we have that

$$\inf_{\Pi} \sum_{e\in\Pi} \lambda^{-|e|} \le |W_n|\lambda^{-n}.$$

There exists a subsequence (n_i) along which the last term tends to zero. $\square$

Furstenberg [3] gave a condition under which branching number and growth do coincide. For $w \in W$, denote by T^w the sub-tree of T comprising w and its descendants, considered as a rooted tree with root w. Let $N \geq 0$. The tree T is called N**-subperiodic** if, for all $w \in W$, there exists w' with $|w'| \leq N$ such that there is an injective graph homomorphism from T^w to $T^{w'}$ mapping w to w'. If T is N-subperiodic for some N, we call T **subperiodic**.

Theorem 3 (Furstenberg [3])**.** *Let T be an infinite, locally finite, rooted tree. If T is subperiodic then $\mathrm{gr}(T)$ exists and equals $\mathrm{br}(T)$.*

For a proof see [3, Prop. III.1] or [10, Sec. 3.3].

Finally in this section we introduce the notions of unimodularity and the mass-transport principle on graphs; more details may be found in [1] and [10, Chap. 8]. Let G be an infinite, locally finite, strongly connected, quasi-transitive directed graph. The stabiliser $\mathrm{Stab}(u)$ of a vertex u is the set of automorphisms that preserve u, and $\mathrm{Stab}(u)v$ denotes the orbit of a vertex v under this set. We may define a positive **weight** function $M : V \to (0, \infty)$ via

$$\frac{M(u)}{M(v)} = \frac{|\,\mathrm{Stab}(u)v|}{|\,\mathrm{Stab}(v)u|}, \qquad u, v \in V, \tag{2.3}$$

where $|\cdot|$ denotes cardinality. The function M is uniquely defined up to multiplication by a constant, and is automorphism-invariant up to multiplication by a constant. The graph G is called **unimodular** if M is constant on each transitivity class. The following fact is very useful.

Theorem 4 (Mass-transport principle)**.** *Let $G = (V, E)$ be an infinite, locally finite, strongly connected, quasi-transitive directed graph with weight function M. Suppose that G is unimodular, and let S be a set comprising a representative from each transitivity class of G. If $m : V \times V \to [0, \infty]$ satisfies $m(\phi u, \phi v) = m(u, v)$ for all $u, v \in V$ and every automorphism ϕ of G, then*

$$\sum_{\substack{s \in S,\\ v \in V}} M(s)^{-1} m(s, v) = \sum_{\substack{s \in S,\\ v \in V}} M(s)^{-1} m(v, s).$$

For proofs of Theorem 4 and the immediately preceding assertions see e.g. [10, Thm 8.10, Cor. 8.11]. To obtain the above formulation, the results of [10] are applied to the undirected graph G' obtained from G by ignoring edge orientations, with the automorphism group of the directed graph G. Thus the assumption (in Theorem 4) of strong connectivity of G may be weakened to that of connectivity of G', in which case we say that G is **weakly connected**.

3. SAW trees

The proof of Theorem 1 is divided into several parts. We start by establishing two of the inequalities required for part (i).

Lemma 5. *Under the assumptions of Theorem 1, the limits μ^{F}, μ^{B}, μ^{FB} exist and satisfy $\mu = \mu^{\mathrm{F}}$ and $\mu^{\mathrm{B}} = \mu^{\mathrm{FB}}$.*

Proof. It is standard that σ_n satisfies the submultiplicative inequality

$$\sigma_{m+n} \leq \sigma_m \sigma_n,$$

and it is easy to see that the sequences σ_n^{F}, σ_n^{B}, σ_n^{FB} satisfy the same inequality. The existence of the constants μ^{F}, μ^{B}, μ^{FB} follows by the subadditive limit theorem (see, for example, [10, Ex. 3.9]).

Fix $v \in V$. From G we construct the rooted **SAW tree** $T(v)$ as follows. The vertices of $T(v)$ are the finite SAWs from v, with the trivial walk of length 0 being the root. Two vertices of $T(v)$ are declared adjacent if one walk is an extension of the other by exactly one step.

Let S be a set of vertices comprising one representative of each transitivity class of G, and let

$$T := \bigvee_{s \in S} T(s)$$

be the rooted tree obtained from disjoint copies of the trees $T(s)$ by joining their roots to one additional vertex o, which is designated the root of the resulting tree. (In the case of transitive G, the argument may be simplified by instead taking $T = T(v)$ for any fixed v.) The level set W_n of T has size

$$|W_n| = \sum_{s \in S} \sigma_{n-1}(s),$$

and hence,

$$\mu = \liminf_{n \to \infty} \sigma_{n-1}^{1/n} \leq \underline{\mathrm{gr}}(T) \leq \overline{\mathrm{gr}}(T) \leq \limsup_{n \to \infty} (|S| \sigma_n)^{1/n} = \mu$$

so that $\mathrm{gr}(T) = \mu$.

Define $T^{\mathrm{F}}(v)$ to be the **forward SAW tree** constructed in an identical manner to $T(v)$ from all forward-extendable SAWs on G from v. Observe that $T^{\mathrm{F}}(v)$ is precisely the tree obtained from $T(v)$ by removing all finite bushes, i.e., removing (together with its incident edges) each vertex w whose rooted subtree $T(v)^w$ is finite. We similarly define $T^{\mathrm{F}} = \bigvee_{s \in S} T^{\mathrm{F}}(s)$, and note that by the above argument, $\mathrm{gr}(T^{\mathrm{F}}) = \mu^{\mathrm{F}}$.

Since both T and T^{F} are 1-subperiodic, Theorem 3 applies to give

$$\mathrm{br}(T) = \mathrm{gr}(T), \qquad \mathrm{br}(T^{\mathrm{F}}) = \mathrm{gr}(T^{\mathrm{F}}).$$

On the other hand, since branching number is unaffected by the removal of finite bushes (by the definition of branching number or by (2.2)), we have $\mathrm{br}(T) = \mathrm{br}(T^{\mathrm{F}})$. Thus $\mu = \mu^{\mathrm{F}}$.

An identical argument gives the equality $\mu^{\mathrm{B}} = \mu^{\mathrm{FB}}$: removing all finite bushes from the **backward SAW tree** gives precisely the **doubly extendable SAW tree** (where these objects are defined by obvious analogy with the previous cases). Thus the two trees have equal branching numbers, whence by Theorem 3 they have equal growths. $\square$

Our next proof employs similar methods.

Proof of Theorem 1(ii). As in the proof of Lemma 5 above, let $T^{\mathrm{F}} := \bigvee_{s\in S} T^{\mathrm{F}}(s)$, where $T^{\mathrm{F}}(v)$ is the forward SAW tree from v and S is a set of representatives of the transitivity classes of G. As argued in the previous proof we have $\mathrm{br}(T^{\mathrm{F}}) = \mathrm{gr}(T^{\mathrm{F}}) = \mu^{\mathrm{F}}$. By the definition of branching number (or (2.2)),

$$\mathrm{br}(T^{\mathrm{F}}) = \max_{s\in S} \mathrm{br}(T^{\mathrm{F}}(s)). \tag{3.1}$$

Therefore there exists $t \in S$ such that $\mathrm{br}(T^{\mathrm{F}}(t)) = \mu^{\mathrm{F}}$.

For every vertex v of G we have $\mathrm{br}(T^{\mathrm{F}}(v)) \le \mu^{\mathrm{F}}$. Call v **good** if equality holds, or **bad** if the inequality is strict. We showed above that good vertices exist. We will see that in fact there are no bad vertices.

For any good vertex u, construct the 'pruned' tree $\widehat{T}^{\mathrm{F}}(u)$ from $T^{\mathrm{F}}(u)$ by removing the subtree $T^{\mathrm{F}}(u)^w$ rooted at each vertex w of $T^{\mathrm{F}}(u)$ that corresponds to a bad vertex of G (i.e., that represents a walk from u ending at a bad vertex). Since each removed subtree has branching number less than $\mu^{\mathrm{F}} - \epsilon$ for some fixed $\epsilon > 0$ depending only on the graph, we have (by the definition of branching number or (2.2)) that

$$\mathrm{br}(\widehat{T}^{\mathrm{F}}(u)) = \mathrm{br}(T^{\mathrm{F}}(u)) = \mu^{\mathrm{F}}.$$

By Lemma 2,

$$\underline{\mathrm{gr}}(\widehat{T}^{\mathrm{F}}(u)) \ge \mathrm{br}(\widehat{T}^{\mathrm{F}}(u)) = \mu^{\mathrm{F}}. \tag{3.2}$$

Let $\widehat{G}$ be the subgraph of G induced by the set of all good vertices, and observe that $\widehat{T}^{\mathrm{F}}(u)$ is precisely the forward SAW tree from u on $\widehat{G}$. Thus (3.2) gives that for any good u,

$$\liminf_{n\to\infty} \sigma_n^{\mathrm{F}}(u, \widehat{G})^{1/n} \ge \mu^{\mathrm{F}}. \tag{3.3}$$

Finally, from any vertex v of G there exists a finite directed walk to some good vertex. Take such a walk of minimum length, say length d and ending at u. Then

$$\sigma_n^{\mathrm{F}}(v, G) \ge \sigma_{n-d}^{\mathrm{F}}(u, \widehat{G}),$$

so by (3.3) we have $\liminf \sigma_n^{\mathrm{F}}(v)^{1/n} \ge \mu^{\mathrm{F}}$. Since by definition of μ^{F} we have $\limsup \sigma_n^{\mathrm{F}}(v)^{1/n} \le \mu^{\mathrm{F}}$, the result follows. $\square$

Alternative proof of Hammersley's result (1.2). The above proof goes through with each $T^{\mathrm{F}}(s)$ replaced by the ordinary SAW tree $T(s)$, and with μ^{F} replaced by μ. $\square$

4. The unimodular case

The proof of the remaining equality of Theorem 1(i) is further divided into two cases according to whether or not the graph is unimodular. In the former case, a stronger statement holds. Let $\overleftarrow{G}$ be the directed graph obtained by reversing all edges of G. Recall that $\sigma_n^{\mathrm{F}}(G) := \sup_{v\in V} \sigma_n^{\mathrm{F}}(v, G)$, and similarly for σ_n^{B}.

Lemma 6. *Under the assumptions of Theorem 1, suppose in addition that G is unimodular. There exists $C = C(G) \geq 1$ such that*

$$C^{-1} \leq \frac{\sigma_n^{\mathrm{F}}(G)}{\sigma_n^{\mathrm{B}}(\overleftarrow{G})} \leq C, \qquad n \geq 0.$$

If in addition G is transitive then we may take $C = 1$.

Proof. Let S be a set of representatives of the transitivity classes of G, and let M be the weight function as in (2.3). There exists $c \geq 1$ such that $c^{-1} \leq M(s)/M(s') \leq c$ for all $s, s' \in S$, with $c = 1$ in the transitive case. By Theorem 4 with $m(u, v)$ defined to be the number of length-n forward-extendable walks from u to v on G,

$$\sum_{s\in S} M(s)^{-1}\sigma_n^{\mathrm{F}}(s, G) = \sum_{s\in S} M(s)^{-1}\sigma_n^{\mathrm{B}}(s, \overleftarrow{G}).$$

We deduce the claimed inequalities with $C = c|S|$. $\square$

If G is an *undirected* unimodular graph (where as usual we interpret an undirected edge as a pair of edges with opposite orientations), then $\overleftarrow{G}$ and G are isomorphic, so Lemma 6 immediately gives $\mu^{\mathrm{B}} = \mu^{\mathrm{F}}$, establishing Theorem 1 in this case.

A directed graph G need not be isomorphic to $\overleftarrow{G}$: an infinite, transitive, unimodular counterexample is given in Figure 1. (A finite counterexample may be obtained by orienting the snub cube in a similar manner.) Nonetheless, we obtain a simple proof of Theorem 1 in the unimodular case, as follows.

Proof of Theorem 1(i), unimodular case. Suppose G is a unimodular graph. By Lemma 6, $\mu^{\mathrm{F}}(G) = \mu^{\mathrm{B}}(\overleftarrow{G})$, so by Lemma 5,

$$\mu^{\mathrm{B}}(G) \leq \mu(G) = \mu^{\mathrm{F}}(G) = \mu^{\mathrm{B}}(\overleftarrow{G}).$$

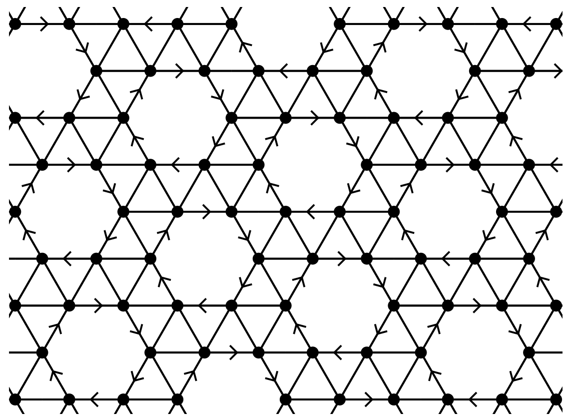

FIGURE 1. An infinite, transitive, directed graph G that is not isomorphic to its edge-reversal $\overleftarrow{G}$. An undirected edge is interpreted as a pair of edges with opposite orientations.

We apply the same argument to $\overleftarrow{G}$, noting that $\overleftarrow{\overleftarrow{G}} = G$, to obtain $\mu^{\mathrm{B}}(\overleftarrow{G}) \leq \mu^{\mathrm{B}}(G)$, so that equality holds throughout. Combined with Lemma 5, this concludes the proof. $\square$

Remark. The assumption of strong connectivity is optional for a unimodular graph G. The above proof of Theorem 1(i) is valid if G is weakly connected in the sense explained after the statement of Theorem 4. If G is not even weakly connected, the same conclusion holds for each weakly connected component of G, and hence for G also.

5. Geodesics

By Lemma 5, $\mu = \mu^{\mathrm{F}}$ and $\mu^{\mathrm{B}} = \mu^{\mathrm{FB}}$, and all that remains is to prove the missing equality in the non-unimodular case. Before doing this we present a proof in the simpler case when G is undirected. This proof applies to both unimodular and non-unimodular undirected graphs, but the subsequent proof for the directed case requires non-unimodularity.

In an undirected graph G, a singly infinite walk with vertex sequence $(v_i)_{i\geq 0}$ is called a **geodesic** if for all $i, j \geq 0$, the graph-distance between v_i and v_j is $|i-j|$. By a standard compactness argument (see, for example, [11, Thm 3.1]), in any infinite, locally finite, connected, undirected graph there is a geodesic starting from any given vertex.

Proof of Theorem 1(i) for undirected graphs. Let G be an undirected graph, and let $v \in V$. Fix a geodesic $\gamma = (v_i)_{i\geq 0}$ started at v. For a SAW w of length n from v, let L be the largest integer for which v_L

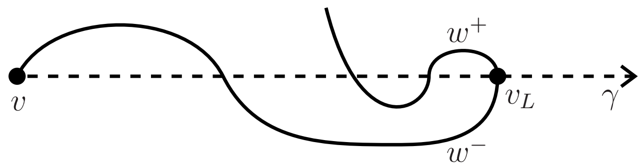

FIGURE 2. The proof of $\mu \le \mu^{\mathrm{B}}$ for undirected graphs: both portions w^-, w^+ of the walk w (solid) are backward extendable via the geodesic γ (dashed).

lies on w. Let w^- and w^+ be the portions of w from v_L to v (reversed), and from v_L to the endpoint of w, respectively. See Figure 2. Both w^- and w^+ are backward extendable via the sub-walk $(v_L, v_{L+1}, \dots)$ of γ. Since γ is a geodesic, we have that $L \le |w^-| \le n$. Therefore,

$$\sigma_n(v) \le \sum_{L=0}^{n} \sum_{k=L}^{n} \sigma_k^{\mathrm{B}}(v_L) \sigma_{n-k}^{\mathrm{B}}(v_L),$$

where k represents $|w^-|$. By the definition of μ^{B}, for every $\epsilon > 0$ there exists $C = C(\epsilon) < \infty$ such that $\sigma_n^{\mathrm{B}} \le C(\mu^{\mathrm{B}} + \epsilon)^n$, and therefore

$$\sigma_n(v) \le C^2(n+1)^2(\mu^{\mathrm{B}} + \epsilon)^n, \qquad n \ge 1,$$

so that $\mu \le \mu^{\mathrm{B}}$. Clearly $\mu^{\mathrm{B}} \le \mu$, so combining with Lemma 5 gives the result. □

Returning to the directed case, we will use the following concept. For $\alpha > 0$, an α-**quasi-geodesic** of a directed graph G is a doubly infinite sequence of vertices $(v_i) = (v_i)_{i \in \mathbb{Z}}$ satisfying

$$d_G(v_i, v_j) \ge \alpha |i - j| \qquad \text{for all } i, j \in \mathbb{Z},$$

where d_G denotes *undirected* graph-distance on G (that is, the length of a shortest path with edges directed arbitrarily). Note that an α-quasi-geodesic is necessarily self-avoiding. A sequence $(v_i)_{i \in \mathbb{Z}}$ is a **quasi-geodesic** if it is an α-quasi-geodesic for some $\alpha > 0$.

Lemma 7. *Suppose the assumptions of Theorem 1 hold and in addition G is not unimodular. There exists a quasi-geodesic $(v_i)_{i \in \mathbb{Z}}$ such that there are directed edges from v_{i+1} to v_i and from v_{-i-1} to v_{-i} for each $i \ge 0$.*

Proof. Since the automorphism group acts quasi-transitively on edges, there exists $C \ge 1$ such that the weight function M satisfies

$$C^{-1} \le \frac{M(u)}{M(v)} \le C, \qquad \langle u, v \rangle \in E. \tag{5.1}$$

Since G is non-unimodular, we may find two vertices u_0, u_1 in the same transitivity class with unequal weights. Assume without loss of generality that $M(u_0) = 1$, and that $c := M(u_1)$ satisfies $c > 1$. Let ϕ be an automorphism mapping u_0 to u_1, and define $u_i = \phi^i(u_0)$ for $i \in \mathbb{Z}$. Since M is automorphism-invariant up to a multiplicative constant,

$$M(u_i) = c^i, \qquad i \in \mathbb{Z}. \tag{5.2}$$

Let ξ be the vertex-sequence of a shortest directed walk from u_1 to u_0 (which exists since G is assumed strongly connected), and let ζ be the vertex-sequence of a shortest directed walk from u_{-1} to u_0. Let $\overline{\xi}$ denote the sequence ξ in reverse order. Let $w = (w_i)_{i\in\mathbb{Z}}$ be the doubly infinite sequence of vertices obtained by concatenating the sequences

$$\ldots, \phi^{-2}\zeta, \phi^{-1}\zeta, \zeta, \overline{\xi}, \phi^1\overline{\xi}, \phi^2\overline{\xi}, \ldots$$

in this order (indexed so that $w_0 = u_0$, and omitting the duplicate vertex where two concatenated sequences meet). Then w forms a doubly infinite path with its edges directed towards w_0, as required for the claimed quasi-geodesic, but it is not necessarily self-avoiding. By (5.1) and (5.2) and the fact that the concatenated walks are bounded in length, there exist $\beta, \gamma > 0$ such that

$$d_G(w_i, w_j) \geq \beta|i-j| - \gamma, \qquad i, j \in \mathbb{Z}. \tag{5.3}$$

We now erase loops from w until we obtain a self-avoiding sequence. More precisely, if there exist $a < b$ with $w_a = w_b$, then choose such a, b with $|a| + |b|$ minimal (say), and remove $w_{a+1}, \ldots, w_{b-1}, w_b$ from the sequence. Iterate this indefinitely. Since initially w visited each vertex only finitely many times, the sequence $(v_i)_{i\in\mathbb{Z}}$ of vertices that are never removed is a self-avoiding sequence. This sequence may furthermore be indexed so that it still has edges directed towards v_0. (We re-index after each loop-erasure: if the chosen loop $w_a, \ldots, w_b$ does not contain w_0, we preserve the index of w_0; if the loop contains w_0, we re-index so that the old w_a becomes the new w_0.) Since loop-erasure does not increase distances along the walk among the vertices that remain, (5.3) holds with (v_i) in place of (w_i) and the same β, γ. Since $(v_i)_{i\in\mathbb{Z}}$ is self-avoiding we may now adjust β so that this inequality holds with $\gamma = 0$. □

Proof of Theorem 1(i), non-unimodular case. Let G be non-unimodular. By Lemma 5, it suffices to prove $\mu = \mu^{\mathrm{B}}$, and obviously we have $\mu^{\mathrm{B}} \leq \mu$. Fix an α-quasi-geodesic $(v_i)_{i\in\mathbb{Z}}$ as in Lemma 7 for some $\alpha > 0$. Let $w = (w_0, \ldots, w_n)$ be the vertex-sequence of an n-step directed SAW starting at $w_0 = v_0$. We will bound the number of such walks above in terms of μ^{B} by considering various cases. Let S_+ be

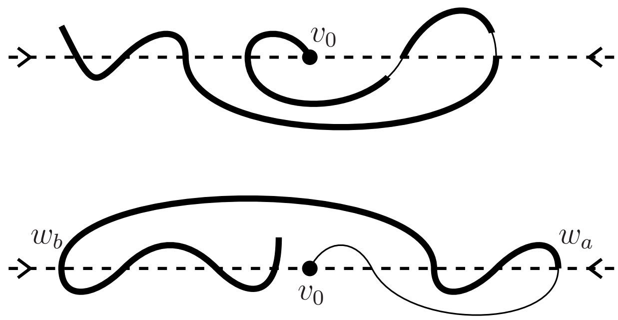

FIGURE 3. Two cases in the proof of $\mu \leq \mu^{\mathrm{B}}$ for undirected non-unimodular graphs: (i) the walk w (solid) has few intersections with the right half of the quasi-geodesic (dashed); (ii) w has many intersections with both halves of the quasi-geodesic. In both cases, each thickened portion of w is backward extendable.

the set of intersections of w with $\{v_i : i \geq 0\}$, and S_- the set of intersections of w with $\{v_i : i \leq 0\}$. Note that, if $v_i \in S_+ \cup S_-$ then $|i| \leq n/\alpha$.

Let $\delta \in (0, \frac{1}{2})$. First suppose that $|S_+| \leq \delta n$. Decompose the walk w into minimal segments starting and ending with an element of S_+, together with (possibly) a final segment starting in S_+. For each such segment $w_a, \ldots, w_b$, its truncation $w_a, \ldots, w_{b-1}$ is backward extendable via the SAW $\ldots, v_{i+2}, v_{i+1}, v_i$, where $v_i = w_a$. See Figure 3(i).

If $|S_-| \leq \delta n$ then the walk may similarly be decomposed to give at most δn backward-extendable segments.

Now suppose that $|S_+|, |S_-| > \delta n$. Let $I_+ = \max\{i : v_i \in S_+\}$ and $I_- = \max\{i : v_{-i} \in S_-\}$, and observe that $I_+, I_- > \delta n$. Thus if we write $v_{I_+} = w_a$ and $v_{-I_-} = w_b$ then

$$|a - b| \geq \alpha(I_+ + I_-) > 2\alpha\delta n,$$

since (v_i) is an α-quasi-geodesic. Writing $m = \min\{a, b\}$, we deduce that $w_m, w_{m+1}, \ldots, w_n$ is a backward-extendable SAW of length greater than $2\alpha\delta n$. See Figure 3(ii).

Combining the various cases, we obtain

$$\sigma_n(v_0) \le 2 \sum_{k\in[0,\delta n]} \binom{\lfloor n/\alpha \rfloor}{k} (2\Delta)^k \sum_{\substack{j_1,\dots,j_k\ge 1:\\ j_1+\cdots+j_k=n}} \sigma^{\mathrm{B}}_{j_1-1}\cdots\sigma^{\mathrm{B}}_{j_k-1} + 2\lfloor n/\alpha\rfloor \sum_{j\in[2\alpha\delta n,n]} \sigma^{\mathrm{B}}_j \sigma_{n-j}, \tag{5.4}$$

where Δ denotes the maximum degree of G. Here the first factor of 2 reflects the two cases $|S_+|\le \delta n$ and $|S_-|\le \delta n$, the integer k is $|S_+|$ or $|S_-|$, the binomial coefficient gives the number of choices for S_+ or S_- as a subset of the quasi-geodesic, and the factor $(2\Delta)^k$ accounts for the choices of directions of segments along the geodesic and of the omitted edges (w_{b-1}, w_b). In the second term, the factor $2\lfloor n/\alpha\rfloor$ bounds the possible choices of the vertex w_m, and $j = n - m$.

Inequality (5.4) implies that $\mu \le \mu^{\mathrm{B}}$, as required. To check this, assume on the contrary that $\mu^{\mathrm{B}} < \mu$. For any $\epsilon > 0$, there exists $C = C(\epsilon) > 0$ such that

$$\sigma^{\mathrm{B}}_n \le C(\mu^{\mathrm{B}}+\epsilon)^n, \qquad \sigma_n \le C(\mu+\epsilon)^n.$$

Substituting this into (5.4), we obtain that

$$\sigma_n(v_0) \le C'n\binom{n/\alpha}{\delta n}\binom{n}{\delta n}\left[(\mu^{\mathrm{B}}+\epsilon)(2\Delta)^{\delta}\right]^n + C''n\left[(\mu^{\mathrm{B}}+\epsilon)^{2\alpha\delta}(\mu+\epsilon)^{1-2\alpha\delta}\right]^n$$

where C', C'' may depend on α and ϵ, and the integer-part symbols in the binomial coefficients have been suppressed to simplify the notation. Therefore, by (1.2),

$$\mu \le \max\left\{(\mu^{\mathrm{B}}+\epsilon)f(\delta), (\mu^{\mathrm{B}}+\epsilon)^{2\alpha\delta}(\mu+\epsilon)^{1-2\alpha\delta}\right\},$$

for some $f(\delta)$ satisfying $f(\delta)\downarrow 1$ as $\delta \downarrow 0$. Let $\epsilon\downarrow 0$. Since $\mu/\mu^{\mathrm{B}} > 1$ by assumption, this is a contradiction for small positive δ. □

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(GRG) Statistical Laboratory, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WB, UK

E-mail address: g.r.grimmett@statslab.cam.ac.uk

URL: http://www.statslab.cam.ac.uk/~grg/

(AEH) Microsoft Research, 1 Microsoft Way, Redmond, WA 98052, USA

E-mail address: holroyd at microsoft.com

URL: http://research.microsoft.com/~holroyd/

(YP) Microsoft Research, 1 Microsoft Way, Redmond, WA 98052, USA

E-mail address: peres@microsoft.com

URL: http://research.microsoft.com/~peres/